

Euler-Poincaré Dynamics of Perfect Complex Fluids

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Abstract

Lagrangian reduction by stages is used to derive the Euler-Poincaré equations for the nondissipative coupled motion and micromotion of complex fluids. We mainly treat perfect complex fluids (PCFs) whose order parameters are *continuous material variables*. These order parameters may be regarded geometrically either as objects in a vector space, or as coset spaces of Lie symmetry groups with respect to subgroups that leave these objects invariant. Examples include liquid crystals, superfluids, Yang-Mills magnetofluids and spin-glasses. A Lie-Poisson Hamiltonian formulation of the dynamics for perfect complex fluids is obtained by Legendre transforming the Euler-Poincaré formulation. These dynamics are also derived by using the Clebsch approach. In the Hamiltonian and Lagrangian formulations of perfect complex fluid dynamics Lie algebras containing two-cocycles arise as a characteristic feature.

After discussing these geometrical formulations of the dynamics of perfect complex fluids, we give an example of how to introduce defects into the order parameter as imperfections (e.g., vortices) that carry their own momentum. The defects may move relative to the Lagrangian fluid material and thereby produce additional reactive forces and stresses.

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1 Introduction

Definitions. The hydrodynamic motion of a complex fluid depends on variables called *order parameters* that describe the macroscopic variations of the internal structure of the fluid parcels. These macroscopic variations may form observable patterns, as seen, for example, via the gradients of optical scattering properties in liquid crystals arising due to the spatially varying orientations of their molecules, as discussed in, e.g., Chandrasekhar [1992] and de Gennes and Prost [1993]. This *micro-order* of a complex fluid is described by an auxiliary macroscopic continuum field of geometrical objects associated with each fluid element and taking values in a vector space (or a manifold) called the *order parameter space*. The canonical example is the description of the local directional asymmetries of nematic liquid crystal molecules by a spatially and temporally varying macroscopic continuum field of unsigned unit vectors called “directors,” see, e.g., Chandrasekhar [1992] and de Gennes and Prost [1993].

Thus, the presence of micro-order breaks the symmetry group \mathcal{O} of the uniform fluid state to a subgroup \mathcal{P} . This *spontaneous symmetry breaking* occurs in every phase transition of condensed matter into an ordered state. The symmetry subgroup $\mathcal{P} \subset \mathcal{O}$ that remains is the isotropy group of whatever geometrical object appears (e.g., a vector, a spin, a director, etc.) when the symmetries of the uniform fluid state are broken to produce the micro-order. That is, the remaining symmetry subgroup \mathcal{P} is the symmetry group of the micro-order. Equivalently, the order parameter may also be regarded as taking its values in the coset space $\mathcal{C} = \mathcal{O}/\mathcal{P}$ and its space and time variations may be represented by a space and time dependent curve in the Lie symmetry group \mathcal{O} through the action of \mathcal{O} on its coset space \mathcal{C} . Thus, associating a geometrical object in a vector space (say, a director) with an order parameter is a way of visualizing the coset space \mathcal{C} , see, e.g., Mermin [1979] and Volovick [1992] for physical examples. The symmetry group \mathcal{O} of the

original uniform fluid state is called the **order parameter group**, or the **broken symmetry**. The order parameter dynamics that generates a curve parameterized by space and time in the broken symmetry group for a complex fluid is called its **micromotion**, although it refers to continuum properties at the coarse-grained macroscopic scale.

Spatial and temporal variations in the micro-order are measured relative to a **reference configuration**. Let \mathcal{O} act transitively from the right on a manifold M . Suppose the subgroup $\mathcal{P} \subset \mathcal{O}$ leaves invariant an arbitrarily chosen reference point $m_0 \in M$, i.e., $m_0 p = m_0, \forall p \in \mathcal{P}$. The reference point m_0 then corresponds to the coset $[e] = e\mathcal{P} = \mathcal{P}$ of \mathcal{O}/\mathcal{P} , where e is the identity element of \mathcal{O} . For another choice of reference point, say $m'_0 = m_0 h$ with $h \in \mathcal{O}$, the isotropy subgroup becomes the conjugate subgroup $\mathcal{P} \rightarrow \mathcal{P}' = h^{-1}\mathcal{P}h$ and the corresponding coset space \mathcal{O}/\mathcal{P}' is, thus, isomorphic to the original coset space \mathcal{O}/\mathcal{P} . Hence, the conjugacy equivalence classes of the order parameter coset spaces account for the arbitrariness in the choice of reference point m_0 . One may think equally well of the order parameter group as acting either on a manifold M with “origin” m_0 , or on a coset space \mathcal{O}/\mathcal{P} , where \mathcal{P} is the stabilizer of the reference point m_0 .

For example, let \mathcal{O} be the group of proper orthogonal transformations $SO(3)$ acting transitively on directors in \mathbf{R}^3 (unit vectors with ends identified) and choose m_0 to be the vertical director in an arbitrary reference frame; so m_0 is invariant under the $O(2)$ group of rotations around the vertical axis in that frame and reflections across its horizontal plane. This example applies to cylindrically symmetric nematic liquid crystal properties. The isotropy group \mathcal{P} is $O(2)$, and the liquid crystal director may be represented in the coset space $SO(3)/O(2)$, which is isomorphic to S^2/Z_2 , the unit sphere S^2 with diametrically opposite points identified, i.e., the real projective plane RP^2 . Thus, after a reference configuration has been chosen, the micro-order of nematic liquid crystals may be represented equivalently as a space and time dependent curve in the broken symmetry group \mathcal{O} acting on either the order parameter manifold $M = RP^2$, or on the coset space $SO(3)/O(2)$ of the broken symmetry $SO(3)$.

In addition to its micromotion, the **motion** of a complex fluid also possesses the usual properties of classical fluid dynamics. In particular, the motion of a complex fluid involves the advection of thermodynamic state variables such as heat and mass, regarded as fluid properties taking values in a vector space V^* . The complete dynamical equations for complex fluids must describe both their motion and their micromotion. In general, these two types of motion will be nonlinearly coupled to each other. This coupling typically arises because properties of both types of motion appear in the stress tensor governing the total momentum.

This definition of complex fluid motion in terms of the continuum dynamics of both its order parameter and its usual fluid properties encompasses a wide range of models for complex fluid motion, including binary fluids, multi-phase fluids, polymeric materials, spin glasses, various other types of magnetic materials, superfluids and, of course, liquid crystals. The order parameter for each of these models provides a continuum (i.e., coarse-grained) description of the complex fluid’s internal degrees of freedom, or micro-order. Here we will discuss mainly the case in which these order parameters are **continuous material variables**, that is, they are continuous functions carried along with the fluid parcels.¹ Such media shall be called **perfect complex fluids**. This name (abbreviated PCF) is chosen to contrast with the perfect simple fluid (which has no internal micro-degrees of freedom) and to provide a geometrical basis for treating defects as imperfections in

¹In some cases, order parameters are determined from constraint relations that are Eulerian in nature, e.g., the volume fraction in two-phase interpenetrating flow as in Holm and Kupersmidt [1986]. This case will not be discussed here.

the order parameter field that can propagate, or move relative to the material labels. The relative motion of defects through the medium generally introduces additional reactive forces and sources of dissipation. We believe the dynamics of defects in complex fluids is best approached after first discussing the dynamics of perfect complex fluids, which is interesting in its own right. Here we shall concentrate on deriving the nonlinear dynamical equations for the ideal continuum motion and micromotion of perfect complex fluids.

Once the equations for the nonlinear dynamics of their ideal (nondissipative) motion and micromotion are established, dissipative processes must be included for most physical applications of complex fluids. By tradition, this is accomplished phenomenologically in these models, by introducing kinetic coefficients, such as viscosity, mobility, thermal diffusivity, etc., so as to obey the requirements of the Clausius-Duhem relation that the entropy production rate be positive when the dynamics of *all thermodynamic variables* (including order parameters) are included, as in Dunn and Serrin [1985], Hohenberg and Halperin [1977]. Here we shall ignore dissipation entirely, trusting that it can be added later by using the standard phenomenological methods.

For the case that the order parameter group \mathcal{O} is the proper orthogonal group $SO(3)$, a geometrical approach to complex fluids exists as part of the rational theory of continuum dynamics for materials with orientational internal degrees of freedom, such as liquid crystals. Rational theories of such complex media began with E. and F. Cosserat [1909]. The Cosserat theories were recapitulated at various times by many different people. See, e.g., Kleman [1983] and Eringen [1997] for descriptions of recent developments and proposed applications of this approach for treating, e.g., liquid crystal dynamics in the tradition of the rational theory of continuum media.

The present paper starts with the example of the Ericksen-Leslie theory of nematic liquid crystals and develops the geometrical framework for continuum theories of perfect complex fluids. In this geometrical framework, the motion and micromotion are nonlinearly and self-consistently coupled to one another by the composite actions of the diffeomorphisms and the order parameter group. The micromotion follows a curve in the order parameter group depending on time and material coordinate, and the motion is a time-dependent curve in the group of diffeomorphisms, which acts on the material coordinates of the fluid parcels to carry them from their reference configuration to their current positions. A feedback develops between the composite motion and micromotion, because the stress tensor affecting the velocity of the diffeomorphisms depends on the gradient of the order parameter. The mathematical basis for our development is the method of **Lagrangian reduction by stages**, due to Cendra, Marsden and Ratiu [1999]. As we shall see, obtaining the Euler-Poincaré equations for perfect complex fluids requires two stages of Lagrangian reduction, first by the order parameter group and then by the diffeomorphism group.

The main results in this paper. The **Euler-Poincaré approach** provides a unified framework for modeling the dynamics of perfect complex fluids that preserves and extends the mathematical structure inherent in the dynamics of classical fluids and liquid crystals in the Eulerian description. This paper provides detailed derivations and discusses applications of the Euler-Lagrange equations, the Lagrange-Poincaré equations, the Euler-Poincaré equations, the Clebsch equations and the Lie-Poisson Hamiltonian structure of perfect complex fluids. The action principles for these equations appear at their various stages of transformation under reduction by symmetries. The theme of this paper is the reduction of degrees of freedom by transforming to variables in the action principles that are invariant under the symmetries of the Lagrangian. (This is the Lagrangian version of Poisson reduction on the Hamiltonian side.) The transformation is done in stages, not all at once,

for the additional perspective we hope it brings and to illustrate how Lagrangian reduction can be iterated in condensed matter applications. It may also be possible to impose nonholonomic constraints by stages, by transforming to variables that either respect the constraints, or appear in their specifications.

The main new results for liquid crystals and PCFs in this paper are:

1. Four action principles for PCF dynamics and their associated motion and micromotion equations at the various stages of reduction – the Euler-Lagrange equations, the Lagrange-Poincaré equations, the Euler-Poincaré equations, and the Clebsch equations.
2. The canonical and Lie-Poisson Hamiltonian formulations of these equations, their Lie algebraic structures and the Poisson maps between them.
3. The momentum conservation laws and Kelvin-Noether circulation theorems for these equations.
4. The reduced equations for one-dimensional dependence on either space or time, and the relation of these reduced equations to the Euler-Poisson equations for the dynamics of generalized tops.
5. A strategy for composing the dynamics of defects in a complex fluid with its underlying PCF dynamics.

Outline. In Section 2, we develop these results for the motion and micromotion in the example of nematic liquid crystals, in forms that shall parallel the results of the general theory derived in the following section. The Euler-Lagrange equations from Hamilton’s principle specialize to the Ericksen-Leslie equations, upon making the appropriate choices of the kinetic and potential energies. The Lagrange-Poincaré equations and Euler-Poincaré equations that follow from applying two successive stages of Lagrangian reduction of Hamilton’s principle with respect to its symmetries provide geometrical variants and generalizations of the Ericksen-Leslie equations.

In Section 3, we perform two successive stages of Lagrangian symmetry reduction to derive first the Lagrange-Poincaré equations and then the Euler-Poincaré equations for PCFs with an arbitrary order parameter group. These Euler-Poincaré equations are Legendre-transformed to their Lie-Poisson Hamiltonian form. Their derivation by the Clebsch approach is also given.

Finally, in Section 4 we discuss a strategy of including defect dynamics into the theory of PCF dynamics. This strategy introduces an additional set of fluid variables that describe the motion of the defects relative to the material coordinates of the PCF. We shall illustrate this strategy in the example of rotating superfluid ^4He , in which $U(1)$ is the broken symmetry and the defects are quantum vortices.

2 The example of liquid crystals

2.1 Background for liquid crystals

We begin with a hands-on example that illustrates the utility of the ideas we shall develop in this paper. Liquid crystals provide a ubiquitous application that embodies these ideas and supplies a guide for developing them.

Liquid crystals are the prototype for complex fluids. For extensive reviews, see Chandrasekhar [1992] and de Gennes and Prost [1993]. An orientational order parameter for a molecule of arbitrary shape is given in Chandrasekhar [1992], p.40, as

$$S = \frac{1}{2} \left\langle 3 \chi \otimes \chi - Id \otimes Id \right\rangle, \quad (2.1)$$

where $\langle \cdot \rangle$ is a statistical average and $\chi \in SO(3)$ is a rotation that specifies the local molecular orientation relative to a fixed reference frame. Thus, in index notation,

$$S_{klKL} = \frac{1}{2} \left\langle 3 \chi_{kK} \chi_{lL} - \delta_{kl} \delta_{KL} \right\rangle. \quad (2.2)$$

This order parameter is traceless in both pairs of its indices, since $\chi^T = \chi^{-1}$.

For cylindrically shaped molecules, (e.g., nematics, cholesterics or smectics²) we may choose the 3-axis, say, as the reference axis of symmetry, i.e., choose $K = 3 = L$ and set

$$n_k \equiv \chi_{k3}, \quad \text{so that} \quad |\mathbf{n}|^2 = \chi_{3k} \chi_{k3} = 1, \quad (2.3)$$

and S becomes,

$$S_{kl33} \equiv S_{kl} = \frac{1}{2} \left\langle 3 n_k n_l - \delta_{kl} \right\rangle. \quad (2.4)$$

Note: The order parameter S does not distinguish between \mathbf{n} and $-\mathbf{n}$. Physically, S may be regarded as the quadrupole moment of a local molecular charge distribution. For a clear description of the use of this order parameter in assessing nematic order-disorder phase transitions using the modern theory of critical phenomena, see Lammert et al. [1995]. Instead of considering such phase transitions, here we shall be interested in the continuum dynamics associated with the interaction of this order parameter with material deformations.

In passing to a continuum mechanics description, one replaces statistical averages by a local space and time dependent unit vector, or “director” $\mathbf{n}(\mathbf{x}, t)$. Then, the continuum order parameter S corresponding to the statistical quantity S in (2.4) is the symmetric traceless tensor,

$$S_{kl}(\mathbf{n}) \equiv \frac{1}{2} \left(3 n_k n_l - \delta_{kl} \right), \quad (2.5)$$

which satisfies $S(\mathbf{n}) = S(-\mathbf{n})$ and admits \mathbf{n} as an eigenvector, $S \cdot \mathbf{n} = \mathbf{n}$. The hydrodynamic tensor order parameter S represents the deviation from isotropy of any convenient tensor property of the medium. For example, the residual dielectric and diamagnetic energy densities of a nematic liquid crystal due to anisotropy may be expressed in terms of the tensor order parameter S as,

$$\frac{\Delta\epsilon}{2} \mathbf{E} \cdot S(\mathbf{n}) \cdot \mathbf{E} \quad \text{and} \quad \frac{\Delta\mu}{2} \mathbf{B} \cdot S(\mathbf{n}) \cdot \mathbf{B}, \quad (2.6)$$

for (external) electric and magnetic fields \mathbf{E} and \mathbf{B} , respectively. (For simplicity, we neglect any dependence of the electric and magnetic polarizabilities of the medium on the gradients $\nabla \mathbf{n}$, although this possibility is allowed.)

²Smectics form layers, so besides their director orientation they have an additional order parameter for their broken translational symmetry.

2.2 Four action principles for perfect liquid crystals

Liquid crystal action. The standard equations for the continuum dynamics of liquid crystals without defects are the Ericksen-Leslie equations, due to Ericksen [1960, 1961], Leslie [1966, 1968], and reviewed, e.g., in Leslie [1979]. These equations express the dynamics of the director \mathbf{n} and may be derived from an action principle $\delta\mathcal{S} = 0$ with action $\mathcal{S} = \int dt L$ in the following class,

$$\mathcal{S} = \int dt \int d^3X \mathcal{L}(\dot{\mathbf{x}}, J, \mathbf{n}, \dot{\mathbf{n}}, \nabla \mathbf{n}), \quad (2.7)$$

with notation $\dot{\mathbf{x}} = \partial \mathbf{x}(\mathbf{X}, t) / \partial t$, so that overdot denotes material time derivative, $J = \det(\partial \mathbf{x} / \partial \mathbf{X})$ and $\nabla \mathbf{n}(\mathbf{X}, t)$ has spatial components given by the chain rule expression,

$$\nabla_i \mathbf{n} = \left[\left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}} \right)^{-1} \right]_{iA} \frac{\partial \mathbf{n}}{\partial X_A} \equiv \frac{\partial X_A}{\partial x_i} \frac{\partial}{\partial X_A} \mathbf{n}(\mathbf{X}, t) \equiv \mathbf{n}_{,i}. \quad (2.8)$$

Note that the coupling between the fluid dynamics $\mathbf{x}(\mathbf{X}, t)$ and director dynamics $\mathbf{n}(\mathbf{X}, t)$ occurs in the Lagrangian density \mathcal{L} through the *inverse deformation gradient*, $(\partial \mathbf{x} / \partial \mathbf{X})^{-1}$, via the chain rule expression above for $\nabla_i \mathbf{n}(\mathbf{X}, t)$.

Varying the action \mathcal{S} in the fields \mathbf{x} and \mathbf{n} at fixed material position \mathbf{X} and time t gives

$$\begin{aligned} \delta\mathcal{S} = - \int dt \int d^3X \left\{ \delta x_p \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_p} \right)^{\cdot} + J \frac{\partial}{\partial x_p} \frac{\partial \mathcal{L}}{\partial J} - J \frac{\partial}{\partial x_m} \left(J^{-1} \frac{\partial \mathcal{L}}{\partial \mathbf{n}_{,m}} \cdot \mathbf{n}_{,p} \right) \right] \right. \\ \left. + \delta \mathbf{n} \cdot \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{n}}} \right)^{\cdot} - \frac{\partial \mathcal{L}}{\partial \mathbf{n}} + J \frac{\partial}{\partial x_m} \left(J^{-1} \frac{\partial \mathcal{L}}{\partial \mathbf{n}_{,m}} \right) \right] \right\}, \end{aligned} \quad (2.9)$$

with natural (homogeneous) conditions, expressing continuity of

$$\frac{\partial \mathcal{L}}{\partial J} \quad \text{and} \quad \hat{n}_m \frac{\partial \mathcal{L}}{\partial \mathbf{n}_{,m}}, \quad (2.10)$$

on a boundary, or at a material interface, whose unit normal has spatial Cartesian components \hat{n}_m . (We always sum over repeated indices.)

Action principle #1 – Euler-Lagrange equations. Stationarity of the action, $\delta\mathcal{S} = 0$, i.e., *Hamilton's principle*, thus yields the following *Euler-Lagrange equations* in the Lagrangian fluid description,

$$\delta x_p : \quad \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_p} \right)^{\cdot} + J \frac{\partial}{\partial x_p} \frac{\partial \mathcal{L}}{\partial J} - J \frac{\partial}{\partial x_m} \left(J^{-1} \frac{\partial \mathcal{L}}{\partial \mathbf{n}_{,m}} \cdot \mathbf{n}_{,p} \right) = 0, \quad (2.11)$$

$$\delta \mathbf{n} : \quad \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{n}}} \right)^{\cdot} - \frac{\partial \mathcal{L}}{\partial \mathbf{n}} + J \frac{\partial}{\partial x_m} \left(J^{-1} \frac{\partial \mathcal{L}}{\partial \mathbf{n}_{,m}} \right) = 0. \quad (2.12)$$

The material volume element $J = \det(\partial \mathbf{x} / \partial \mathbf{X})$ satisfies an auxiliary kinematic equation, obtained from its definition,

$$j = J \frac{\partial X_A}{\partial x_i} \frac{\partial \dot{x}_i}{\partial X_A}. \quad (2.13)$$

Imposing constant J , say $J = 1$, gives incompressible flow. The last term in equation (2.11) is the divergence of the Ericksen stress tensor arising due to the dependence of the potential energy of the medium on the strain $\nabla \mathbf{n}$, see Ericksen [1960, 1961].

With the proper choice of Lagrangian density, namely,

$$\mathcal{L} = \frac{\rho}{2}|\dot{\mathbf{x}}|^2 + \frac{I}{2}|\dot{\mathbf{n}}|^2 + p(J - 1) + q(|\mathbf{n}|^2 - 1) - F(\mathbf{n}, \nabla \mathbf{n}), \quad (2.14)$$

where ρ and I are material constants one finds that the Euler-Lagrange equations (2.11) and (2.12) produce the **Ericksen-Leslie equations** for incompressible liquid crystal flow, Chandrasekhar [1992] and de Gennes and Prost [1993],

$$\rho \ddot{x}_i + \frac{\partial}{\partial x_j} \left(p \delta_{ij} + \mathbf{n}_{,i} \cdot \frac{\partial F}{\partial \mathbf{n}_{,j}} \right) = 0, \quad (2.15)$$

$$I \ddot{\mathbf{n}} - 2q\mathbf{n} + \mathbf{h} = 0, \quad \text{with} \quad \mathbf{h} = \frac{\partial F}{\partial \mathbf{n}} - \frac{\partial}{\partial x_j} \frac{\partial F}{\partial \mathbf{n}_{,j}}. \quad (2.16)$$

In the Lagrangian density (2.14), the Lagrange multipliers p and q enforce the incompressibility condition $J = 1$ and the director normalization condition $|\mathbf{n}|^2 = 1$, respectively. The standard choice for the function $F(\mathbf{n}, \nabla \mathbf{n})$ is the Oseen-Zöcher-Frank Helmholtz free energy density, as discussed, e.g., in Chandrasekhar [1992] and de Gennes and Prost [1993], namely,³

$$\begin{aligned} F(\mathbf{n}, \nabla \mathbf{n}) &= \underbrace{\frac{k_2(\mathbf{n} \cdot \nabla \times \mathbf{n})}{2}}_{\text{chirality}} + \underbrace{\frac{k_{11}(\nabla \cdot \mathbf{n})^2}{2}}_{\text{splay}} \\ &+ \underbrace{\frac{k_{22}(\mathbf{n} \cdot \nabla \times \mathbf{n})^2}{2}}_{\text{twist}} + \underbrace{\frac{k_{33}|\mathbf{n} \times \nabla \times \mathbf{n}|^2}{2}}_{\text{bend}} \\ &+ \frac{\Delta\epsilon}{2} \mathbf{E} \cdot S(\mathbf{n}) \cdot \mathbf{E} + \frac{\Delta\mu}{2} \mathbf{B} \cdot S(\mathbf{n}) \cdot \mathbf{B}, \end{aligned} \quad (2.17)$$

with $k_2 = 0$ for nematics (but nonzero for cholesterics) and $\mathbf{n} \cdot \nabla \times \mathbf{n} = 0$ for smectics. Since smectics form layers that break translational symmetry, their order parameter group may be taken as the Euclidean group $E(3)$, or perhaps as $SO(3) \times U(1)$ in simple cases. We shall concentrate our attention on nematics in this section. However, the general theory for arbitrary order parameter groups developed in the next section would also encompass smectics.

Key concepts: Material angular frequency and spatial strain of rotation. The rate of rotation of a director \mathbf{n} in the rest frame of the fluid material element that carries it is given by

$$\boldsymbol{\nu} = \mathbf{n} \times \dot{\mathbf{n}}. \quad (2.18)$$

In terms of this **material angular frequency** $\boldsymbol{\nu}$ (which is orthogonal to the director \mathbf{n}), the dynamical equation (2.16) becomes simply,

$$I \dot{\boldsymbol{\nu}} = \mathbf{h} \times \mathbf{n}. \quad (2.19)$$

³This notation treats spatial and director components on the same (Cartesian) basis. Later, we will distinguish between these types of components, see, e.g., equation (2.82).

Likewise, the amount by which a specified director field $\mathbf{n}(\mathbf{x})$ rotates under an infinitesimal spatial displacement from x_i to $x_i + dx_i$ is given by,

$$\gamma_i = \mathbf{n} \times \nabla_i \mathbf{n} \quad \text{or,} \quad \gamma \cdot dx \equiv \gamma_i dx_i = \mathbf{n} \times d\mathbf{n}. \quad (2.20)$$

Remarks on kinematics. An orthogonal matrix with components χ_{kL} is associated with an orthonormal frame of unit vectors $(\mathbf{l}, \mathbf{m}, \mathbf{n})$ whose components $(l_k, m_k, n_k) = (\chi_{k1}, \chi_{k2}, \chi_{k3})$ satisfy

$$\epsilon_{jkl}(d\chi\chi^{-1})_{lk} = (\mathbf{l} \times d\mathbf{l} + \mathbf{m} \times d\mathbf{m} + \mathbf{n} \times d\mathbf{n})_j, \quad (2.21)$$

which is $O(3)$ right invariant. Accordingly, the spatial rotational strain, with components

$$(\mathbf{n} \times d\mathbf{n})_j = \epsilon_{jkl}(d\chi_{l3}\chi_{k3}), \quad (2.22)$$

and the material angular velocity,

$$(\mathbf{n} \times \dot{\mathbf{n}})_j = \epsilon_{jkl}(\dot{\chi}_{l3}\chi_{k3}), \quad (2.23)$$

are each elements of TS^2/Z_2 , i.e., they are tangent to the sphere $|\mathbf{n}|^2 = 1$ and $O(2)$ right invariant under $SO(2)$ rotations about \mathbf{n} and under reflections $\mathbf{n} \rightarrow -\mathbf{n}$. Also, $\mathbf{n} \times d\mathbf{n}$ and $\mathbf{n} \times \dot{\mathbf{n}}$ are $O(2)$ right invariant elements of the Lie algebra, $so(3) \simeq TSO(3)/SO(3)$, so they are elements of $so(3)/O(2)$. Being orthogonal to \mathbf{n} , their evolution preserves $|\mathbf{n}|^2$. Therefore, $|\mathbf{n}|^2 = 1$ may be taken as an initial condition, and \mathbf{n} may be reconstructed from $\dot{\mathbf{n}} = \boldsymbol{\nu} \times \mathbf{n}$.

The **spatial rotational strain** $\mathbf{n} \times d\mathbf{n} = \gamma \cdot dx$ satisfies the kinematic relation,

$$(\gamma \cdot dx)^\bullet = (2\boldsymbol{\nu} \times \gamma_i + \nabla_i \boldsymbol{\nu}) dx_i, \quad (2.24)$$

obtained from its definition. This kinematic relation involves only γ_i , $\boldsymbol{\nu}$ and x_i , which suggests transforming variables from $(\mathbf{n}, \dot{\mathbf{n}}, \nabla \mathbf{n})$ to γ_i and $\boldsymbol{\nu}$. (Note that γ_i and $\boldsymbol{\nu}$ do not distinguish between \mathbf{n} and $-\mathbf{n}$.) Each component of γ_i is orthogonal to the director \mathbf{n} . That is, $\mathbf{n} \cdot \gamma_i = 0$, for $i = 1, 2, 3$. Using this fact and $|\mathbf{n}|^2 = 1$ gives the relations,

$$\nabla_i \mathbf{n} = -\mathbf{n} \times \gamma_i \quad \text{and} \quad \nabla \times \mathbf{n} = -\mathbf{n} \operatorname{tr} \gamma + n^m \gamma_m, \quad (2.25)$$

or, equivalently, in index notation,

$$\nabla_i n^a = -\epsilon^{abc} n^b \gamma_i^c \quad \text{and} \quad \epsilon^{ijk} \nabla_j n^k = -n^i \gamma_m^m + n^m \gamma_m^i, \quad (2.26)$$

where lower indices are spatial and upper indices denote director components. Consequently, we have the following useful **transformation formulas**,

$$\nabla_i \mathbf{n} \cdot \nabla_j \mathbf{n} = \gamma_i \cdot \gamma_j, \quad \nabla_i \mathbf{n} \times \nabla_j \mathbf{n} = \gamma_i \times \gamma_j \quad \text{and} \quad \mathbf{n} \cdot \nabla \times \mathbf{n} = -\operatorname{tr} \gamma. \quad (2.27)$$

Connection, curvature, singularities and topological indices. The Eulerian curl of the spatial rotational strain γ_i yields the remarkable relation,

$$B_{ij} \equiv \gamma_{i,j} - \gamma_{j,i} + 2\gamma_i \times \gamma_j = 0, \quad (2.28)$$

where B_{ij} vanishes, provided the vector field \mathbf{n} is continuous. In geometrical terms, vanishing of B_{ij} is the Maurer-Cartan relation for the flat connection one-form (or, left-invariant Cartan one-form) $\gamma_i dx_i = \mathbf{n} \times d\mathbf{n}$, as discussed in, e.g., Flanders [1989]. That is, the curvature two-form $B_{ij} dx_i \wedge dx_j$ vanishes in the absence of singularities (disclinations) in the director field $\mathbf{n}(\mathbf{x})$. Thus, B_{ij} may be regarded as the areal density of disclinations in a nematic liquid crystal. In Section 3, we shall discuss how to proceed when defects exist and the disclination density B_{ij} does not vanish.

A second remarkable formula also stems from the curl- γ equation (2.28), which implies the relation,

$$\frac{\partial \gamma_j^a}{\partial x_i} dx_i \wedge dx_j = \epsilon^{abc} \frac{\partial n^b}{\partial x_i} \frac{\partial n^c}{\partial x_j} dx_i \wedge dx_j = \epsilon^{abc} dn^b \wedge dn^c, \quad (2.29)$$

for a continuous director field. Contracting with n^a and taking the exterior derivative of this relation implies

$$\frac{\partial}{\partial x_k} \left(n^a \frac{\partial \gamma_j^a}{\partial x_i} \right) dx_i \wedge dx_j \wedge dx_k = \epsilon^{abc} dn^a \wedge dn^b \wedge dn^c = \det(\nabla n) d^3 x = \det(\gamma) d^3 x. \quad (2.30)$$

By $|\mathbf{n}|^2 = 1$, these expressions must vanish, when the director field has no singularities. Thus, in the absence of singularities, there is a vector \mathbf{v} , for which

$$\epsilon_{kij} n^a \frac{\partial \gamma_j^a}{\partial x_i} = \epsilon_{kij} \epsilon^{abc} n^a \frac{\partial n^b}{\partial x_i} \frac{\partial n^c}{\partial x_j} = \epsilon_{kij} v_{j,i} = (\text{curl } \mathbf{v})_k, \quad (2.31)$$

or, equivalently,

$$\epsilon^{abc} n^a \nabla n^b \times \nabla n^c = \text{curl } \mathbf{v}. \quad (2.32)$$

Up to an inessential multiplicative factor, this formula is the well known **Mermin-Ho relation** between three-dimensional superfluid texture \mathbf{n} and vorticity $\text{curl } \mathbf{v}$ in dipole-locked $^3\text{He}-A$, found in Mermin and Ho [1976]. Since \mathbf{n} is a unit vector, a simple calculation in spherical polar coordinates transforms this relation to

$$\epsilon^{abc} n^a \nabla n^b \times \nabla n^c = \nabla \phi \times \nabla \cos \theta = \text{curl } \mathbf{v}, \quad (2.33)$$

for $\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)^T$ in terms of the polar angle θ and azimuthal angle ϕ on the sphere. Hence,

$$\text{curl } \mathbf{v} \cdot d\mathbf{S} = \epsilon^{abc} n^a dn^b \wedge dn^c = d\phi \wedge d\cos \theta, \quad (2.34)$$

which is the area element on the sphere. Thus, by Stokes' theorem for a given closed curve C in \mathbf{R}^3 with unit tangent vector $\mathbf{n}(\mathbf{x})$, the integral

$$2\pi W_r \equiv \oint_C \mathbf{v} \cdot d\mathbf{x} = \iint_S \epsilon^{abc} n^a \gamma_k^b dx_k \wedge \gamma_l^c dx_l, \quad (2.35)$$

is the (signed) area swept out by \mathbf{n} on a portion of the unit sphere upon traversing the curve C which is the boundary of the surface S in three dimensions. This is also equal to the *writhe* of the curve C , see Fuller [1978], as cited in Kamien [1998]. Thus, if the field lines of the three dimensional vector field $\mathbf{n}(\mathbf{x})$ form a closed curve C , the writhe of this curve is given by the area integral in equation (2.35) over a surface whose boundary is C . Equation (2.35) is the second remarkable formula implied by the curl- γ equation (2.28).

Finally, the integral of $\mathbf{v} \cdot \text{curl } \mathbf{v}$ over three dimensional space gives a number,

$$N = \frac{1}{4\pi} \int_{S^3} \mathbf{v} \cdot \text{curl } \mathbf{v} \, d^3x, \quad (2.36)$$

which takes integer values and is called the **Hopf index** or degree of mapping for the map $\mathbf{n}(\mathbf{x}) : S^3 \rightarrow S^2$. The Hopf degree of mapping N counts the number of times the map $\mathbf{n}(\mathbf{x}) : S^3 \rightarrow S^2$ covers the unit sphere, as discussed in, e.g., Flanders [1989]. Equivalently, the Hopf degree of mapping N counts the number of linkages of the three dimensional divergenceless vector field $\text{curl } \mathbf{v}$ with itself. See Mineev [1980] for the physical interpretation of this formula in terms of disclinations in nematic liquid crystal experiments. See also Trebin [1982], Kleman [1983], Kleman [1989] and references therein for additional discussions of the differential geometry of defects in liquid crystal physics.

The quantity $\epsilon^{abc} n^a \nabla n^b \times \nabla n^c$ in equation (2.33) appears in many differential geometric contexts in physics: in the Mermin-Ho relation for the vorticity of superfluid $^3\text{He}-A$ in terms of the “texture” \mathbf{n} due to Mermin and Ho [1976]; in the Skyrminion Lagrangian, as discussed in, e.g., Trebin [1982]; as the n -field topological Wess-Zumino term in the $O(3)$ nonlinear sigma model, discussed in, e.g., Yabu and Kuratsuji [1999], Tsurumaru [1999]; and as the instanton number density in coset models. For references to the last topic see Coquereaux and Jadcyk [1994] and for a discussion of the associated Poisson-Lie models, see Stern [1999]. This same term also produces forces on vortices in $^3\text{He} - A(B)$ as found in Hall [1985] and in ferromagnets, discussed in Kuratsuji and Yabu [1998]. Also, we have seen that this term also allows the calculation of the “writhe” of a self-avoiding closed loop of, say, DNA, or some other polymer filament with unit tangent vector \mathbf{n} , see, e.g., Fuller [1978], Goldstein et al. [1998], Kamien [1998]. See also Klapper [1996] and Goriely and Tabor [1997], for additional recent discussions of the differential geometric writhe of DNA conformations. Finally, we have identified the relation of the Hopf degree of mapping to the linkage number of the *writhe flux*, $\text{curl } \mathbf{v}$. Perhaps surprisingly, in the case of liquid crystals, it will turn out that this topological index, the writhe of a fluid loop, and the flux of curvature B_{ij} through a fluid surface (which measures the presence of singularities in the director field) are all three **created** nonabelian PCF dynamics.

Action principle #2 – Lagrange-Poincaré equations. Intrigued by the geometrical properties of the variables γ_i and ν , we shall seek to express the Ericksen-Leslie equations solely in terms of these variables, by applying symmetry reduction methods to Hamilton’s principle for liquid crystal dynamics. The trace of the spatial rotational strain γ gives

$$\gamma_m^a \equiv \epsilon^{abc} n^b \nabla_m n^c \quad \Rightarrow \quad \text{tr } \gamma \equiv \gamma_m^m = -\mathbf{n} \cdot \nabla \times \mathbf{n}, \quad (2.37)$$

which figures in the energies of chirality and twist in equation (2.17) for $F(\mathbf{n}, \nabla \mathbf{n})$.

We shall now drop the anisotropic dielectric and diamagnetic terms, which exert torques on the director angular momentum due to external \mathbf{E} and \mathbf{B} fields given by,

$$\mathbf{h}_{EM} \times \mathbf{n} = \frac{3\Delta\epsilon}{2}(\mathbf{E} \cdot \mathbf{n})\mathbf{E} \times \mathbf{n} + \frac{3\Delta\mu}{2}(\mathbf{B} \cdot \mathbf{n})\mathbf{B} \times \mathbf{n}, \quad (2.38)$$

but which do not contribute in the motion equation. Upon neglecting these torques, which may always be restored later (see the Appendix), the remaining Oseen-Frank free energy $F(\mathbf{n}, \nabla \mathbf{n})$ may be expressed (modulo a divergence term that will not contribute to the Euler-Lagrange equations) entirely as a function of a subset of the invariants of γ , namely, $\text{tr}\gamma$, $\text{tr}(\gamma + \gamma^T)^2$ and $\text{tr}(\gamma - \gamma^T)^2$, with the help of the identities below, obtained using $|\mathbf{n}|^2 = 1$. In the notation of Eringen [1997], one has

$$I_1 \equiv (\nabla \times \mathbf{n})^2 = [\mathbf{n} \cdot (\nabla \times \mathbf{n})]^2 + [\mathbf{n} \times (\nabla \times \mathbf{n})]^2, \quad (2.39)$$

$$\begin{aligned} I_2 &\equiv n_{,j}^i n_{,j}^i = (\nabla \cdot \mathbf{n})^2 + \left(\nabla \times \mathbf{n} \right)^2 + \left(n_{,j}^j n_{,j}^i - n_{,i}^j n_{,j}^i \right)_{,i} \\ &= \left(\nabla \times \mathbf{n} \right)^2 + n_{,i}^j n_{,j}^i, \end{aligned} \quad (2.40)$$

$$I_3 \equiv [(\mathbf{n} \cdot \nabla) \mathbf{n}]^2 = [\mathbf{n} \times (\nabla \times \mathbf{n})]^2. \quad (2.41)$$

Geometrically, the divergence term

$$\left(n_{,j}^j n_{,j}^i - n_{,i}^j n_{,j}^i \right)_{,i} = 2\kappa_1 \kappa_2, \quad (2.42)$$

appearing in equation (2.40) for I_2 is (twice) the Gaussian curvature of the surface Σ whose normal is $\mathbf{n}(\mathbf{x})$. This equation follows from Rodrigues' formula

$$d\mathbf{n} = -\kappa d\mathbf{x}, \quad (2.43)$$

for the principle directions of the surface Σ . Such a surface exists globally, provided $\mathbf{n} \cdot \text{curl } \mathbf{n} = 0$, see Weatherburn [1974], as cited in Kleman [1983]. Straightforward calculations give the following linear relations, see Eringen [1997]

$$\text{tr}(\gamma_S^2) = -\frac{1}{2} I_1 + I_2 + \frac{1}{2} I_3, \quad \text{tr}(\gamma_A^2) = -\frac{1}{2} (I_1 + I_3), \quad (2.44)$$

where γ_S and γ_A are the symmetric and antisymmetric parts of γ ,

$$\gamma_S = \frac{1}{2}(\gamma + \gamma^T), \quad \text{and} \quad \gamma_A = \frac{1}{2}(\gamma - \gamma^T). \quad (2.45)$$

Hence, modulo the divergence term in (2.40), the Oseen-Frank free energy in (2.17) may be written in terms of the invariants of γ as $F(\text{tr}\gamma, \text{tr}(\gamma_S^2), \text{tr}(\gamma_A^2))$.

The reduced Lagrange-Poincaré action. By transforming to the rotational strain γ_m and frequency ν , the action \mathcal{S} for liquid crystals in (2.7) may thus be *reduced*, that is, rewritten in fewer variables, by defining,

$$\mathcal{S} = \int dt \int d^3 X \mathcal{L}(\dot{\mathbf{x}}, J, \nu, \gamma_m), \quad (2.46)$$

where the reduced variables $\boldsymbol{\nu}$ and $\boldsymbol{\gamma}_m$ are perpendicular to the director \mathbf{n} .

A calculation using their definitions shows that the variations of J and $\boldsymbol{\nu}$ satisfy

$$\delta J = J \frac{\partial X_A}{\partial x_i} \frac{\partial \delta x_i}{\partial X_A}, \quad \delta \boldsymbol{\nu} = (\mathbf{n} \times \delta \mathbf{n})^\cdot - 2\boldsymbol{\nu} \times (\mathbf{n} \times \delta \mathbf{n}), \quad (2.47)$$

in terms of the variational quantity $\mathbf{n} \times \delta \mathbf{n}$. Likewise, we calculate the variation of the rotational strain $\boldsymbol{\gamma}_m$ as

$$\begin{aligned} \delta \boldsymbol{\gamma}_m &= J^{-1} \frac{\partial}{\partial X_A} \left[J \frac{\partial X_A}{\partial x_m} (\mathbf{n} \times \delta \mathbf{n}) \right] \\ &\quad - 2\boldsymbol{\gamma}_m \times (\mathbf{n} \times \delta \mathbf{n}) - \boldsymbol{\gamma}_p J^{-1} \frac{\partial}{\partial X_A} \left[J \frac{\partial X_A}{\partial x_m} \delta x_p \right]. \end{aligned} \quad (2.48)$$

This variational expression for $\boldsymbol{\gamma}_m$ may be rearranged into the more suggestive form,

$$\delta \boldsymbol{\gamma}_m + \boldsymbol{\gamma}_p \nabla_m \delta x_p = 2(\mathbf{n} \times \delta \mathbf{n}) \times \boldsymbol{\gamma}_m + \nabla_m (\mathbf{n} \times \delta \mathbf{n}), \quad (2.49)$$

in which its similarity with the kinematic relation for $\boldsymbol{\gamma}$ in (2.24) is more readily apparent, especially when the kinematic equations (2.13) and (2.24) are rewritten in the notation of equation (2.8) as

$$\dot{J} = J \nabla_i \dot{x}_i \quad \text{and} \quad \dot{\boldsymbol{\gamma}}_m + \boldsymbol{\gamma}_p \nabla_m \dot{x}_p = 2\boldsymbol{\nu} \times \boldsymbol{\gamma}_m + \nabla_m \boldsymbol{\nu}. \quad (2.50)$$

The variation of the Lagrange-Poincaré action \mathcal{S} in (2.46) for liquid crystals in the Lagrangian fluid description may now be rewritten as

$$\begin{aligned} \delta \mathcal{S} &= - \int dt \int d^3 X \left\{ \delta x_p \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_p} \right)^\cdot + J \frac{\partial}{\partial x_p} \frac{\partial \mathcal{L}}{\partial J} - J \frac{\partial}{\partial x_m} \left(J^{-1} \frac{\partial \mathcal{L}}{\partial \boldsymbol{\gamma}_m} \cdot \boldsymbol{\gamma}_p \right) \right] \right. \\ &\quad + (\mathbf{n} \times \delta \mathbf{n}) \cdot \left[\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\nu}} \right)^\cdot - 2\boldsymbol{\nu} \times \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nu}} + J \frac{\partial}{\partial x_m} \left(J^{-1} \frac{\partial \mathcal{L}}{\partial \boldsymbol{\gamma}_m} \right) - 2\boldsymbol{\gamma}_m \times \frac{\partial \mathcal{L}}{\partial \boldsymbol{\gamma}_m} \right] \Big\} \\ &\quad + \int dt \int d^3 X \left\{ \left(\delta x_p \frac{\partial \mathcal{L}}{\partial \dot{x}_p} + (\mathbf{n} \times \delta \mathbf{n}) \cdot \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nu}} \right)^\cdot \right. \\ &\quad \left. + \frac{\partial}{\partial X_A} \left[\frac{\partial X_A}{\partial x_m} \delta x_p \left(J \frac{\partial \mathcal{L}}{\partial J} \delta_{mp} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{\gamma}_m} \cdot \boldsymbol{\gamma}_p \right) + \frac{\partial X_A}{\partial x_m} (\mathbf{n} \times \delta \mathbf{n}) \cdot \frac{\partial \mathcal{L}}{\partial \boldsymbol{\gamma}_m} \right] \right\}. \end{aligned} \quad (2.51)$$

Consequently, the action principle $\delta \mathcal{S} = 0$ yields the following **Lagrange-Poincaré equations** for liquid crystals,

$$\begin{aligned} \delta x_p : \quad & \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_p} \right)^\cdot + J \frac{\partial}{\partial x_p} \frac{\partial \mathcal{L}}{\partial J} - J \frac{\partial}{\partial x_m} \left(J^{-1} \frac{\partial \mathcal{L}}{\partial \boldsymbol{\gamma}_m} \cdot \boldsymbol{\gamma}_p \right) = 0, \\ \mathbf{n} \times \delta \mathbf{n} : \quad & \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\nu}} \right)^\cdot - 2\boldsymbol{\nu} \times \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nu}} + J \frac{\partial}{\partial x_m} \left(J^{-1} \frac{\partial \mathcal{L}}{\partial \boldsymbol{\gamma}_m} \right) - 2\boldsymbol{\gamma}_m \times \frac{\partial \mathcal{L}}{\partial \boldsymbol{\gamma}_m} = 0, \end{aligned} \quad (2.52)$$

with natural boundary conditions, cf. equation (2.10),

$$\partial \mathcal{L} / \partial J = 0 \quad \text{and} \quad \hat{n}_m \frac{\partial \mathcal{L}}{\partial \boldsymbol{\gamma}_m} = 0, \quad (2.53)$$

on a boundary, or material interface, whose unit normal has spatial Cartesian components \hat{n}_m . These are the **dynamical boundary conditions** for liquid crystal motion. These conditions ensure that the fluid pressure $\partial\mathcal{L}/\partial J$ and the normal stress are continuous across a fluid interface. The term $\gamma_p \cdot \partial\mathcal{L}/\partial\gamma_m$ contributes to the stress tensor of the complex fluid and arises from the dependence of its free energy upon the rotational strain γ_i . Again $J = 1$ for incompressible flow, as imposed by the Lagrange multiplier p , the fluid pressure. (The Lagrange multiplier q is no longer necessary.)

Upon specializing to the Oseen-Frank-type Lagrangian (2.14), stationarity of the action $\mathcal{S}(\dot{\mathbf{x}}, J, \boldsymbol{\nu}, \boldsymbol{\gamma})$ in equation (2.46) recovers the Ericksen-Leslie equations (2.15) written in these variables. This is an example of **Lagrangian reduction** – from the variables $(\mathbf{n}, \dot{\mathbf{n}}, \nabla\mathbf{n})$ to $(\boldsymbol{\nu}, \boldsymbol{\gamma})$. The equations in fewer variables that result from the first stage of Lagrangian reduction were named the Lagrange-Poincaré equations in Marsden and Scheurle [1995], Cendra, Marsden and Ratiu [1999]. One of the goals of this paper is to characterize the geometrical properties of such equations for PCFs, using the mathematical framework of Lagrangian reduction by stages established in Cendra, Marsden and Ratiu [1999]. From this viewpoint, the first stage of the Lagrangian reduction for liquid crystals is finished.

Action principle #3 – Euler-Poincaré equations. The Lagrange-Poincaré equations (2.52) and their kinematic relations (2.50) are still expressed in the Lagrangian fluid description. We shall pass to the Eulerian fluid description by applying a second stage of Lagrangian reduction, now defined by the following right actions of the diffeomorphism group,

$$\begin{aligned} u(x, t) &= \dot{x}(X, t)g^{-1}(t), \\ D(x, t)d^3x &= d^3Xg^{-1}(t), \\ \boldsymbol{\nu}(x, t) &= \boldsymbol{\nu}(X, t)g^{-1}(t), \\ \gamma_i(x, t)dx_i &= \left(\gamma_i(X, t) \frac{\partial x_i(X, t)}{\partial X_A} dX_A \right) g^{-1}(t), \end{aligned} \tag{2.54}$$

where the right action denoted $x(X, t) = Xg(t)$ defines the fluid motion as following a time-dependent curve $g(t)$ in the diffeomorphism group as it acts on the reference configuration of the fluid with coordinate X . In the Eulerian fluid description, the action S in Hamilton's principle $\delta S = 0$ is written as

$$\mathcal{S} = \int dt \int d^3x \ell(\mathbf{u}, D, \boldsymbol{\nu}, \boldsymbol{\gamma}), \tag{2.55}$$

in terms of the Lagrangian density ℓ given by

$$\ell(\mathbf{u}, D, \boldsymbol{\nu}, \boldsymbol{\gamma})d^3x = \mathcal{L}\left(\dot{x}g^{-1}(t), Jg^{-1}(t), \boldsymbol{\nu}g^{-1}(t), \boldsymbol{\gamma}g^{-1}(t)\right)\left(d^3Xg^{-1}(t)\right). \tag{2.56}$$

The variations of these Eulerian fluid quantities are computed from their definitions to be,

$$\delta u_j = \frac{\partial \eta_j}{\partial t} + u_k \frac{\partial \eta_j}{\partial x_k} - \eta_k \frac{\partial u_j}{\partial x_k}, \quad (2.57)$$

$$\delta D = -\frac{\partial D \eta_j}{\partial x_j}, \quad (2.58)$$

$$\delta \boldsymbol{\nu} = \frac{\partial \boldsymbol{\Sigma}}{\partial t} + u_m \frac{\partial \boldsymbol{\Sigma}}{\partial x_m} - 2\boldsymbol{\nu} \times \boldsymbol{\Sigma} - \eta_m \frac{\partial \boldsymbol{\nu}}{\partial x_m}, \quad (2.59)$$

$$\delta \boldsymbol{\gamma}_m = \frac{\partial \boldsymbol{\Sigma}}{\partial x_m} - 2\boldsymbol{\gamma}_m \times \boldsymbol{\Sigma} - \eta_k \frac{\partial \boldsymbol{\gamma}_m}{\partial x_k} - \boldsymbol{\gamma}_k \frac{\partial \eta_k}{\partial x_m}, \quad (2.60)$$

where $\boldsymbol{\Sigma}(x, t) \equiv (\mathbf{n} \times \delta \mathbf{n})(X, t)g^{-1}(t)$ and $\eta \equiv \delta g g^{-1}(t)$. One may compare these Eulerian variations with the Lagrangian variations in (2.47) and (2.48). See also Holm, Marsden and Ratiu [1998] for more discussion of such constrained variations in Eulerian fluid dynamics. Here the quantities D and $\boldsymbol{\gamma}_m$ satisfy the following **Eulerian kinematic equations**, also obtained from their definitions, cf. the Lagrangian kinematic relations (2.50),

$$\frac{\partial D}{\partial t} = -\frac{\partial D u_j}{\partial x_j}, \quad (2.61)$$

$$\frac{\partial \boldsymbol{\gamma}_m}{\partial t} = \frac{\partial \boldsymbol{\nu}}{\partial x_m} + 2\boldsymbol{\nu} \times \boldsymbol{\gamma}_m - u_k \frac{\partial \boldsymbol{\gamma}_m}{\partial x_k} - \boldsymbol{\gamma}_k \frac{\partial u_k}{\partial x_m}. \quad (2.62)$$

Note the similarity in form between the Eulerian variations of D and $\boldsymbol{\gamma}_m$, and their corresponding kinematic equations. This similarity arises because both the variations and the evolution equations for these quantities are obtained as infinitesimal Lie group actions.

Remark. The kinematic formula (2.62) for the evolution of $\boldsymbol{\gamma}_m$ immediately implies the following **$\boldsymbol{\gamma}$ -circulation theorem** for the spatial rotational strain,

$$\frac{d}{dt} \oint_{c(u)} \boldsymbol{\gamma}_m dx_m = \oint_{c(u)} 2\boldsymbol{\nu} \times \boldsymbol{\gamma}_m dx_m. \quad (2.63)$$

Thus, the circulation of $\boldsymbol{\gamma}$ around a loop $c(u)$ moving with the fluid is only conserved when $\boldsymbol{\gamma}_m \times \boldsymbol{\nu}$ is a gradient. Otherwise, the curl of $\boldsymbol{\gamma}_m \times \boldsymbol{\nu}$ generates circulation of $\boldsymbol{\gamma}$ around fluid loops. By Stokes' theorem and equation (2.29) we have,

$$\oint_{c(u)} \boldsymbol{\gamma}_m^a dx_m = \int_{S(u)} \frac{\partial \boldsymbol{\gamma}_m^a}{\partial x_j} dx_j \wedge dx_m = \int_{S(u)} \epsilon^{abc} dn^b \wedge dn^c, \quad (2.64)$$

for a surface $S(u)$ whose boundary is the fluid loop $c(u)$. Consequently, the $\boldsymbol{\gamma}$ -circulation theorem (2.63) implies that a nonvanishing curl of $\boldsymbol{\gamma}_m \times \boldsymbol{\nu}$ generates a time-changing flux of $\epsilon^{abc} \nabla n^b \times \nabla n^c$ for $a = 1, 2, 3$, through those surfaces whose boundaries move with the fluid.

The nonabelian 2-cocycle terms create with and linkages. We use the kinematic equation (2.62) for gamma and the definition of $\boldsymbol{\nu}$ to calculate the evolution of $\text{curl} \mathbf{v} \cdot d\mathbf{S} = n^a \epsilon^{abc} dn^b \wedge dn^c$ as

$$(\partial_t + \mathcal{L}_u)(\text{curl} \mathbf{v} \cdot d\mathbf{S}) = 2 d\nu^b \wedge dn^b = d(\mathbf{n} \cdot 2\boldsymbol{\nu} \times \boldsymbol{\gamma}_m dx_m) = d(\mathbf{n} \cdot \text{ad}_\nu \boldsymbol{\gamma}_m dx_m)$$

This means the nonabelian aspect of the generalized 2-cocycle (the $\nabla \boldsymbol{\nu}$ in the gamma equation) **creates writhe** in the boundary of a surface moving with the fluid under the gamma-evolution. That is, by Stokes law,

$$\frac{d}{dt} \oint_{c(u)} \mathbf{v} \cdot d\mathbf{x} = \oint_{c(u)} \mathbf{n} \cdot 2\boldsymbol{\nu} \times \boldsymbol{\gamma}_m dx_m = \oint_{c(u)} \mathbf{n} \cdot \text{ad}_{\boldsymbol{\nu}} \boldsymbol{\gamma}_m dx_m,$$

and the writhe of a fluid loop is not preserved. Likewise, we calculate

$$(\partial_t + \mathcal{L}_u)(\mathbf{v} \cdot d\mathbf{x} \wedge d(\mathbf{v} \cdot d\mathbf{x})) \neq \text{exact form}.$$

Therefore, we find

$$\frac{d}{dt} \int \mathbf{v} \cdot \text{curl } \mathbf{v} d^3x \neq 0.$$

Thus, perhaps surprisingly, the linkage number, the Hopf degree on mapping in equation (2.36) is also not preserved by the gamma-evolution. So the nonabelian generalized 2-cocycle term in the gamma equation also **creates linkages** in the Mermin-Ho quantity, $\text{curl } \mathbf{v}$. Conversely, the preservation of these quantities is an Abelian characteristic.

Euler-Poincaré action variations. We compute the variation of the action (2.55) in Eulerian variables at fixed time t and spatial position \mathbf{x} as,

$$\begin{aligned} \delta \mathcal{S} &= \int dt \int d^3x \left[\frac{\delta \ell}{\delta u_j} \delta u_j + \frac{\delta \ell}{\delta D} \delta D + \frac{\delta \ell}{\delta \boldsymbol{\nu}} \cdot \delta \boldsymbol{\nu} + \frac{\delta \ell}{\delta \boldsymbol{\gamma}_m} \cdot \delta \boldsymbol{\gamma}_m \right] \\ &= \int dt \int d^3x \left\{ \eta_j \left[-\frac{\partial}{\partial t} \frac{\delta \ell}{\delta u_j} - \frac{\delta \ell}{\delta u_k} \frac{\partial u_k}{\partial x_j} - \frac{\partial}{\partial x_k} \left(\frac{\delta \ell}{\delta u_j} u_k \right) + D \frac{\partial}{\partial x_j} \frac{\delta \ell}{\delta D} \right. \right. \\ &\quad \left. \left. - \frac{\delta \ell}{\delta \boldsymbol{\nu}} \cdot \frac{\partial \boldsymbol{\nu}}{\partial x_j} - \frac{\delta \ell}{\delta \boldsymbol{\gamma}_m} \cdot \frac{\partial \boldsymbol{\gamma}_m}{\partial x_j} + \frac{\partial}{\partial x_m} \left(\boldsymbol{\gamma}_j \cdot \frac{\delta \ell}{\delta \boldsymbol{\gamma}_m} \right) \right] \right. \\ &\quad + \boldsymbol{\Sigma} \cdot \left[-\frac{\partial}{\partial t} \frac{\delta \ell}{\delta \boldsymbol{\nu}} - \frac{\partial}{\partial x_m} \left(\frac{\delta \ell}{\delta \boldsymbol{\nu}} u_m + \frac{\delta \ell}{\delta \boldsymbol{\gamma}_m} \right) + 2\boldsymbol{\nu} \times \frac{\delta \ell}{\delta \boldsymbol{\nu}} + 2\boldsymbol{\gamma}_m \times \frac{\delta \ell}{\delta \boldsymbol{\gamma}_m} \right] \\ &\quad \left. + \frac{\partial}{\partial t} \left[\eta_j \frac{\delta \ell}{\delta u_j} + \boldsymbol{\Sigma} \cdot \frac{\delta \ell}{\delta \boldsymbol{\nu}} \right] \right. \\ &\quad \left. + \frac{\partial}{\partial x_m} \left[\eta_j \left(\frac{\delta \ell}{\delta u_j} u_m - D \frac{\delta \ell}{\delta D} \delta_{jm} - \boldsymbol{\gamma}_j \cdot \frac{\delta \ell}{\delta \boldsymbol{\gamma}_m} \right) + \boldsymbol{\Sigma} \cdot \left(\frac{\delta \ell}{\delta \boldsymbol{\nu}} u_m + \frac{\delta \ell}{\delta \boldsymbol{\gamma}_m} \right) \right] \right\}, \end{aligned} \tag{2.65}$$

where we have substituted the variational expressions (2.57)-(2.60) and integrated by parts.

The dynamical equations resulting from Hamilton's principle $\delta \mathcal{S} = 0$ are obtained by requiring the coefficients of the arbitrary variations η_j and $\boldsymbol{\Sigma}$ to vanish. We assume these variations themselves vanish at the temporal endpoints and we defer discussing the boundary terms for a moment.

Hence, we obtain the **Euler-Poincaré equations** for liquid crystals,

$$\eta_j : \quad \frac{\partial}{\partial t} \frac{\delta \ell}{\delta u_j} = - \frac{\delta \ell}{\delta u_k} \frac{\partial u_k}{\partial x_j} - \frac{\partial}{\partial x_k} \left(\frac{\delta \ell}{\delta u_j} u_k \right) + D \frac{\partial}{\partial x_j} \frac{\delta \ell}{\delta D} - \frac{\delta \ell}{\delta \boldsymbol{\nu}} \cdot \frac{\partial \boldsymbol{\nu}}{\partial x_j} - \frac{\delta \ell}{\delta \boldsymbol{\gamma}_m} \cdot \frac{\partial \boldsymbol{\gamma}_m}{\partial x_j} + \frac{\partial}{\partial x_m} \left(\boldsymbol{\gamma}_j \cdot \frac{\delta \ell}{\delta \boldsymbol{\gamma}_m} \right), \quad (2.66)$$

$$\begin{aligned} \boldsymbol{\Sigma} : \quad \frac{\partial}{\partial t} \frac{\delta \ell}{\delta \boldsymbol{\nu}} &= - \frac{\partial}{\partial x_m} \left(\frac{\delta \ell}{\delta \boldsymbol{\nu}} u_m + \frac{\delta \ell}{\delta \boldsymbol{\gamma}_m} \right) \\ &\quad + 2\boldsymbol{\nu} \times \frac{\delta \ell}{\delta \boldsymbol{\nu}} + 2\boldsymbol{\gamma}_m \times \frac{\delta \ell}{\delta \boldsymbol{\gamma}_m}. \end{aligned} \quad (2.67)$$

Equation (2.67) agrees with the Lagrangian version of the micromotion equation governing $\boldsymbol{\nu}$ in (2.52). To see this, it is helpful to recall that $\delta \ell / \delta \boldsymbol{\nu}$ is an Eulerian density, so a Jacobian is involved in the transformation to the Lagrangian version.

Momentum conservation. In momentum conservation form, the liquid crystal motion equation (2.66) in the Eulerian fluid description becomes

$$\frac{\partial}{\partial t} \frac{\delta \ell}{\delta u_j} = - \frac{\partial}{\partial x_m} \left(\frac{\delta \ell}{\delta u_j} u_m + \left(\ell - D \frac{\partial \ell}{\partial D} \right) \delta_{mj} - \boldsymbol{\gamma}_j \cdot \frac{\partial \ell}{\partial \boldsymbol{\gamma}_m} \right), \quad (2.68)$$

for simple algebraic dependence of the Lagrangian ℓ on \mathbf{u} , D , $\boldsymbol{\nu}$ and $\boldsymbol{\gamma}_m$. This momentum conservation law is in agreement with the direct passage to Eulerian coordinates of the motion equation (2.52) in the Lagrangian fluid description. For this transformation, it is helpful to recognize from equation (2.56) that $\ell = (\mathcal{L}J^{-1})g(t)^{-1}$ implies, by the chain rule,

$$\frac{\partial \mathcal{L}}{\partial J} g(t)^{-1} = \ell - D \frac{\partial \ell}{\partial D}, \quad \text{since} \quad D(x, t) = J^{-1}(X, t)g(t)^{-1}. \quad (2.69)$$

The momentum conservation law (2.68) acquires additional terms, if the Lagrangian ℓ also depends on gradients of \mathbf{u} , D , $\boldsymbol{\nu}$ and $\boldsymbol{\gamma}_m$.

Noether's theorem. Noether's theorem associates conservation laws to continuous symmetries of Hamilton's principle. See, e.g., Olver [1993] for a clear discussion of the classical theory and Jackiw and Manton [1980] for its applications in gauge theories. The momentum conservation equation (2.68) also emerges from Noether's theorem, since the action \mathcal{S} in equation (2.55) admits spatial translations, that is, since this action is invariant under the transformations,

$$x_j \rightarrow x'_j = x_j + \eta_j(\mathbf{x}, t) \quad \text{with} \quad \eta_j = c_j, \quad (2.70)$$

for constants c_j , with $j = 1, 2, 3$. To see how equation (2.68) emerges from Noether's theorem, simply add the term $\partial(\ell \eta_j) / \partial x_j$ (arising from transformations of the spatial coordinate) to the endpoint and boundary terms in the variational formula (2.65) arising from variations at *fixed* time t and spatial position \mathbf{x} , then specialize to $\eta_j = c_j$.

Kelvin-Noether circulation theorem for liquid crystals. Rearranging the motion equation (2.66) and using the continuity equation for D in (2.61) gives the **Kelvin-Noether circulation theorem**, cf. Holm, Marsden and Ratiu [1998],

$$\frac{d}{dt} \oint_{c(u)} \frac{1}{D} \frac{\delta \ell}{\delta u_j} dx_j = - \oint_{c(u)} \frac{1}{D} \left[\frac{\delta \ell}{\delta \boldsymbol{\nu}} \cdot d\boldsymbol{\nu} + \frac{\delta \ell}{\delta \boldsymbol{\gamma}_m} \cdot d\boldsymbol{\gamma}_m - \frac{\partial}{\partial x_m} \left(\boldsymbol{\gamma}_j \cdot \frac{\delta \ell}{\delta \boldsymbol{\gamma}_m} \right) dx_j \right]. \quad (2.71)$$

Hence, stresses in the director field of a liquid crystal and gradients in its angular frequency $\boldsymbol{\nu}$ and rotational strain $\boldsymbol{\gamma}$ can generate fluid circulation. Equivalently, by Stokes' theorem, these gradients of liquid crystal properties can generate vorticity, defined as $\boldsymbol{\omega} \equiv \text{curl}(D^{-1}\delta\ell/\delta\mathbf{u})$. For incompressible flows, we may set $D = 1$ in these equations and write the vorticity dynamics as,

$$\begin{aligned} \frac{\partial \omega_i}{\partial t} + (\mathbf{u} \cdot \nabla) \omega_i - (\boldsymbol{\omega} \cdot \nabla) u_i &= \left(\nabla \nu^a \times \nabla \frac{\delta \ell}{\delta \nu^a} \right)_i + \left(\nabla \gamma_m^a \times \nabla \frac{\delta \ell}{\delta \gamma_m^a} \right)_i \\ &\quad + \epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_m} \left(\gamma_k \cdot \frac{\delta \ell}{\delta \boldsymbol{\gamma}_m} \right). \end{aligned} \quad (2.72)$$

Thus, spatial gradients in the director angular frequency $\boldsymbol{\nu}$ and rotational strain $\boldsymbol{\gamma}$ are sources of the fluid vorticity in a liquid crystal.

2.3 Hamiltonian dynamics of perfect liquid crystals

The Euler-Lagrange-Poincaré formulation of liquid crystal dynamics obtained so far allows passage to the corresponding Hamiltonian formulation via the following **Legendre transformation** of the reduced Lagrangian ℓ in the velocities \mathbf{u} and $\boldsymbol{\nu}$, in the Eulerian fluid description,

$$m_i = \frac{\delta \ell}{\delta u_i}, \quad \boldsymbol{\sigma} = \frac{\delta \ell}{\delta \boldsymbol{\nu}}, \quad h(\mathbf{m}, D, \boldsymbol{\sigma}, \boldsymbol{\gamma}_m) = m_i u_i + \boldsymbol{\sigma} \cdot \boldsymbol{\nu} - \ell(\mathbf{u}, D, \boldsymbol{\nu}, \boldsymbol{\gamma}_m). \quad (2.73)$$

Accordingly, one computes the derivatives of h as

$$\frac{\delta h}{\delta m_i} = u_i, \quad \frac{\delta h}{\delta \boldsymbol{\sigma}} = \boldsymbol{\nu}, \quad \frac{\delta h}{\delta D} = - \frac{\delta \ell}{\delta D}, \quad \frac{\delta h}{\delta \boldsymbol{\gamma}_m} = - \frac{\delta \ell}{\delta \boldsymbol{\gamma}_m}. \quad (2.74)$$

Consequently, the Euler-Poincaré equations (2.66) - (2.67) and the auxiliary kinematic equations (2.61) - (2.62) for liquid crystal dynamics in the Eulerian description imply the following equations, for the Legendre-transformed variables, $(\mathbf{m}, D, \boldsymbol{\sigma}, \boldsymbol{\gamma})$,

$$\begin{aligned} \frac{\partial m_i}{\partial t} &= -m_j \frac{\partial}{\partial x_i} \frac{\delta h}{\delta m_j} - \frac{\partial}{\partial x_j} \left(m_i \frac{\delta h}{\delta m_j} \right) - D \frac{\partial}{\partial x_i} \frac{\delta h}{\delta D} \\ &\quad + \left(\frac{\partial \boldsymbol{\gamma}_j}{\partial x_i} \right) \cdot \frac{\delta h}{\delta \boldsymbol{\gamma}_j} - \frac{\partial}{\partial x_j} \left(\boldsymbol{\gamma}_i \cdot \frac{\delta h}{\delta \boldsymbol{\gamma}_j} \right) - \boldsymbol{\sigma} \cdot \frac{\partial}{\partial x_i} \frac{\delta h}{\delta \boldsymbol{\sigma}}, \end{aligned} \quad (2.75)$$

$$\frac{\partial D}{\partial t} = - \frac{\partial}{\partial x_j} \left(D \frac{\delta h}{\delta m_j} \right), \quad (2.76)$$

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} = - \frac{\partial}{\partial x_j} \left(\boldsymbol{\sigma} \frac{\delta h}{\delta m_j} - \frac{\delta h}{\delta \boldsymbol{\gamma}_j} \right) - 2\boldsymbol{\sigma} \times \frac{\delta h}{\delta \boldsymbol{\sigma}} - 2\boldsymbol{\gamma}_j \times \frac{\delta h}{\delta \boldsymbol{\gamma}_j}, \quad (2.77)$$

$$\frac{\partial \boldsymbol{\gamma}_i}{\partial t} = -\boldsymbol{\gamma}_j \frac{\partial}{\partial x_i} \frac{\delta h}{\delta m_j} - \left(\frac{\partial \boldsymbol{\gamma}_i}{\partial x_j} \right) \frac{\delta h}{\delta m_j} + \frac{\partial}{\partial x_i} \frac{\delta h}{\delta \boldsymbol{\sigma}} - 2\boldsymbol{\gamma}_i \times \frac{\delta h}{\delta \boldsymbol{\sigma}}. \quad (2.78)$$

These equations are **Hamiltonian**. That is, they may be expressed in the form

$$\frac{\partial \mathbf{z}}{\partial t} = \{\mathbf{z}, h\} = \mathbf{b} \cdot \frac{\delta h}{\delta \mathbf{z}}, \quad (2.79)$$

where $\mathbf{z} \in (\mathbf{m}, D, \boldsymbol{\sigma}, \boldsymbol{\gamma})$ and the Hamiltonian matrix \mathbf{b} defines the Poisson bracket

$$\{f, h\} = \int d^n x \frac{\delta f}{\delta \mathbf{z}} \cdot \mathbf{b} \cdot \frac{\delta h}{\delta \mathbf{z}}, \quad (2.80)$$

which is bilinear, skew symmetric and satisfies the **Jacobi identity**,

$$\{f, \{g, h\}\} + c.p.(f, g, h) = 0.$$

Assembling the liquid crystal equations (2.75) - (2.78) into the Hamiltonian form (2.79) gives,

$$\frac{\partial}{\partial t} \begin{bmatrix} m_i \\ D \\ \gamma_i \\ \boldsymbol{\sigma} \end{bmatrix} = - \begin{bmatrix} m_j \partial_i + \partial_j m_i & D \partial_i & (\partial_j \gamma_i - \gamma_{j,i}) \cdot & \boldsymbol{\sigma} \cdot \partial_i \\ \partial_j D & 0 & 0 & 0 \\ \gamma_j \partial_i + \gamma_{i,j} & 0 & 0 & -\partial_i + 2\boldsymbol{\gamma}_i \times \\ \partial_j \boldsymbol{\sigma} & 0 & -\partial_j + 2\boldsymbol{\gamma}_j \times & 2\boldsymbol{\sigma} \times \end{bmatrix} \begin{bmatrix} \delta h / \delta m_j \\ \delta h / \delta D \\ \delta h / \delta \gamma_j \\ \delta h / \delta \boldsymbol{\sigma} \end{bmatrix} \quad (2.81)$$

In components, this Hamiltonian matrix expression becomes,

$$\frac{\partial}{\partial t} \begin{bmatrix} m_i \\ D \\ \gamma_i^\alpha \\ \sigma_\alpha \end{bmatrix} = - \begin{bmatrix} m_j \partial_i + \partial_j m_i & D \partial_i & \partial_j \gamma_i^\beta - \gamma_{j,i}^\beta & \sigma_\beta \partial_i \\ \partial_j D & 0 & 0 & 0 \\ \gamma_j^\alpha \partial_i + \gamma_{i,j}^\alpha & 0 & 0 & -\delta_\beta^\alpha \partial_i - t_{\beta\kappa}^\alpha \gamma_i^\kappa \\ \partial_j \sigma_\alpha & 0 & -\delta_\alpha^\beta \partial_j + t_{\alpha\kappa}^\beta \gamma_j^\kappa & -t_{\alpha\beta}^\kappa \sigma_\kappa \end{bmatrix} \begin{bmatrix} \delta h / \delta m_j \\ \delta h / \delta D \\ \delta h / \delta \gamma_j^\beta \\ \delta h / \delta \sigma_\beta \end{bmatrix} \quad (2.82)$$

where $t_{\beta\kappa}^\alpha = 2\epsilon_{\alpha\beta\kappa}$ represents (twice) the vector cross product for liquid crystals. In this expression, the operators act to the right on all terms in a product by the chain rule and, as usual, the summation convention is enforced on repeated indices. At this point we have switched to using both lower and upper Greek indices for the internal degrees of freedom, so that we will agree later with the more general theory, in which upper Greek indices refer to a basis set in a Lie algebra, and lower Greek indices refer to the corresponding dual basis. Lower Latin indices still denote spatial components.

Remarks about the Hamiltonian matrix. The Hamiltonian matrix in equation (2.82) was discovered some time ago in the context of spin-glasses and Yang-Mills magnetohydrodynamics (YM-MHD) by using the Hamiltonian approach in Holm and Kupershmidt [1988]. There, it was shown to be a valid Hamiltonian matrix by associating its Poisson bracket as defined in equation (2.80) with the dual space of a certain Lie algebra of semidirect-product type that has a generalized two-cocycle on it. The mathematical discussion of this Lie algebra and its generalized two-cocycle is given in Holm and Kupershmidt [1988]. A related Poisson bracket for spin glass fluids is given in Volovik and Dotsenko [1980]. A different Poisson bracket for nematic liquid crystals is given in Kats and Lebedev [1994], who discuss a constrained Poisson bracket that in general does not satisfy the Jacobi identity. The liquid crystal equations in Kats and Lebedev [1994] also differ from the Ericksen-Leslie equations by being first order in the time derivatives of the director, rather than second order, as in the Ericksen-Leslie theory. See also Isaev et al. [1995] for a discussion of this first order theory using the Poisson bracket approach. The present work ignores the first order (kinematic) theory in what follows and concentrates on second order (dynamic) theory.

Being dual to a Lie algebra, our matrix in equation (2.82) is in fact a **Lie-Poisson Hamiltonian matrix**. See, e.g., Marsden and Ratiu [1999] and references therein for more discussions of such Hamiltonian matrices. For our present purposes, its rediscovery in the PCF context links the physical and mathematical interpretations of the variables in the theory of PCFs with earlier work in the gauge theory approach to condensed matter, see, e.g., Kleinert [1989]. These gauge theory aspects emerge upon rewriting the Lie-Poisson Hamiltonian equations in terms of **covariant derivatives** with respect to the space-time connection one-form given by $\boldsymbol{\nu}dt + \boldsymbol{\gamma}_m dx_m$, as done in Holm and Kupershmidt [1988]. The gauge theory approach to liquid crystal physics is reviewed in, e.g., Trebin [1982], Kleman [1983], and Kleman [1989].

The generalized two-cocycle in the Hamiltonian matrix (2.82) is somewhat exotic for a classical fluid. This generalized two-cocycle consists of the partial derivatives in equation (2.82) appearing with the Kronecker deltas in the $\boldsymbol{\sigma} - \boldsymbol{\gamma}$ cross terms. The first hint of these terms comes from the exterior derivative $d\boldsymbol{\nu}$ appearing in the kinematic equation (2.62) for $\boldsymbol{\gamma}_m$. Finding such a feature in the continuum theory of liquid crystals may provide a bridge for transferring ideas and technology between the classical and quantum fluid theories.⁴ See Volovick [1992] and Volovick and Vachaspati [1996] for discussions of similar opportunities for technology transfer in the theory of superfluid Helium. The implications of the generalized two-cocycle for the solutions of the liquid crystal equations can be seen by considering two special cases: static solutions with one-dimensional spatial dependence; and spatially homogeneous, but time-dependent, liquid crystal dynamics.

Static perfect liquid crystal solutions with z -variation. Static (steady, zero-velocity) solutions, with one-dimensional spatial variations in, say, the z -direction obey equations (2.75) - (2.78) specialized to

$$\begin{aligned} -D \frac{d}{dz} \frac{\delta h}{\delta D} &= \boldsymbol{\gamma}_3 \cdot \frac{d}{dz} \frac{\delta h}{\delta \boldsymbol{\gamma}_3} + \boldsymbol{\sigma} \cdot \frac{d}{dz} \frac{\delta h}{\delta \boldsymbol{\sigma}} = 0, \\ \frac{d}{dz} \frac{\delta h}{\delta \boldsymbol{\gamma}_3} &= 2\boldsymbol{\gamma}_3 \times \frac{\delta h}{\delta \boldsymbol{\gamma}_3} + 2\boldsymbol{\sigma} \times \frac{\delta h}{\delta \boldsymbol{\sigma}}, \\ \frac{d}{dz} \frac{\delta h}{\delta \boldsymbol{\sigma}} &= 2\boldsymbol{\gamma}_3 \times \frac{\delta h}{\delta \boldsymbol{\sigma}}. \end{aligned} \tag{2.83}$$

The sum of terms in the first equation of the set (2.83) vanishes to give zero pressure gradient, as a consequence of the latter two equations. We compare the latter two equations with the $E(3)$ **Lie-Poisson Hamiltonian systems**, given by

$$\begin{aligned} -\frac{d\boldsymbol{\Pi}}{dt} &= \frac{\partial H}{\partial \boldsymbol{\Pi}} \times \boldsymbol{\Pi} + \frac{\partial H}{\partial \boldsymbol{\Gamma}} \times \boldsymbol{\Gamma}, \\ -\frac{d\boldsymbol{\Gamma}}{dt} &= \frac{\partial H}{\partial \boldsymbol{\Pi}} \times \boldsymbol{\Gamma}. \end{aligned} \tag{2.84}$$

When the Hamiltonian $H(\boldsymbol{\Pi}, \boldsymbol{\Gamma})$ in these equations is specialized to

$$H = \left(\sum_{i=1}^3 \frac{(\Pi_i)^2}{2I_i} \right) + Mg\boldsymbol{\chi} \cdot \boldsymbol{\Gamma}, \tag{2.85}$$

⁴In quantum field theory, these partial derivative operators in the Poisson bracket (or commutator relations) are called non-ultralocal terms, or Schwinger terms, after Schwinger [1951, 1959]. These terms lead to the so-called “quantum anomalies.” Of course, no quantum effects are considered here. However, the Poisson bracket (2.82) still contains *classical* Schwinger terms.

where I_i , $i = 1, 2, 3$, and $Mg\chi$ are constants, then the $E(3)$ Lie-Poisson equations (2.84) specialize to the classical Euler-Poisson equations for a heavy top, discussed in, e.g., Marsden and Ratiu [1999].

By comparing these two equation sets, one observes that (at least for algebraic h) the static perfect liquid crystal flows with z -variation in equations (2.83) and the $E(3)$ Lie-Poisson tops governed by equations (2.84) are **Legendre duals** to each other under the map,

$$\frac{d}{dz} = -2 \frac{d}{dt}, \quad h(\gamma_3, \sigma) = \Pi \cdot \gamma_3 + \Gamma \cdot \sigma - H(\Pi, \Gamma), \quad (2.86)$$

so that

$$\frac{\partial h}{\partial \gamma_3} = \Pi, \quad \frac{\partial h}{\partial \sigma} = \Gamma, \quad \gamma_3 = \frac{\partial H}{\partial \Pi}, \quad \sigma = \frac{\partial H}{\partial \Gamma}. \quad (2.87)$$

Hence, we arrive at the result:

The class of static one dimensional flows of a perfect liquid crystal is Legendre-isomorphic to the class of $E(3)$ Lie-Poisson tops.

These tops conserve $\Pi \cdot \Gamma$ and $|\Gamma|^2$, but in general they are not integrable.

Spatially homogeneous, time-dependent perfect liquid crystal flows. Spatially homogeneous solutions of equations (2.75) - (2.78) obey the dynamical equations,

$$\frac{1}{2} \frac{d\sigma}{dt} = \frac{\delta h}{\delta \sigma} \times \sigma + \frac{\delta h}{\delta \gamma_m} \times \gamma_m, \quad (2.88)$$

$$\frac{1}{2} \frac{d\gamma_m}{dt} = \frac{\delta h}{\delta \sigma} \times \gamma_m. \quad (2.89)$$

For a single value of the spatial index, say $m = 3$, these are nothing more than the $E(3)$ top equations (2.84) with time re-parameterized by $t \rightarrow -2t$. Hence, in this case, the Hamiltonian internal dynamics of a spatially homogeneous liquid crystal is essentially identical to the $E(3)$ Lie-Poisson dynamics of a top. In the multi-component case, one sums over $m = 1, 2, 3$, in the second term of equation (2.88) and, thus, the resulting dynamics is more complex than the simple top. Hence, we have:

The class of spatially homogeneous, time-dependent perfect liquid crystal flows is isomorphic to the generalization (2.88) - (2.89) of the $E(3)$ Lie-Poisson tops.

Action principle #4 – Clebsch representation. Another representation of Hamilton's principle for liquid crystals in the Eulerian fluid description may be found by constraining the Eulerian action S in equation (2.55) by using Lagrange multipliers to enforce the kinematic equations (2.61) and (2.62). The constrained action is, thus,

$$\begin{aligned} \mathcal{S} = & \int dt \int d^3x \ell(\mathbf{u}, D, \boldsymbol{\nu}, \boldsymbol{\gamma}) + \phi \left(\frac{\partial D}{\partial t} + \frac{\partial D u_j}{\partial x_j} \right) \\ & + \beta_m \cdot \left(\frac{\partial \gamma_m}{\partial t} - \frac{\partial \boldsymbol{\nu}}{\partial x_m} + 2 \gamma_m \times \boldsymbol{\nu} + u_k \frac{\partial \gamma_m}{\partial x_k} + \gamma_k \frac{\partial u_k}{\partial x_m} \right), \end{aligned} \quad (2.90)$$

with Lagrange multipliers ϕ and β_m . Stationarity of S under variations in u_k and ν implies the relations

$$\delta u_k : \quad \frac{\delta \ell}{\delta u_k} - D \frac{\partial \phi}{\partial x_k} + \beta_m \cdot \frac{\partial \gamma_m}{\partial x_k} - \frac{\partial}{\partial x_m} (\gamma_k \cdot \beta_m) = 0, \quad (2.91)$$

$$\delta \nu : \quad \frac{\delta \ell}{\delta \nu} + \frac{\partial \beta_m}{\partial x_m} + 2 \beta_m \times \gamma_m = 0. \quad (2.92)$$

These are the **Clebsch relations** for the momentum of the motion $m_k = \delta \ell / \delta u_k$ and the director angular momentum of the micromotion $\sigma = \delta \ell / \delta \nu$. Stationary variations of S in D and γ_m give, respectively, the dynamical equations for the **canonical momenta**, $\pi_D = \phi$ and $\pi_{\gamma_m} = \beta_m$, as

$$\delta D : \quad \frac{\partial \phi}{\partial t} + u_k \frac{\partial \phi}{\partial x_k} - \frac{\delta \ell}{\delta D} = 0, \quad (2.93)$$

$$\delta \gamma_m : \quad \frac{\partial \beta_m}{\partial t} + \frac{\partial}{\partial x_k} (\beta_m u_k) - \beta_k \frac{\partial u_m}{\partial x_k} - 2 \nu \times \beta_m - \frac{\delta \ell}{\delta \gamma_m} = 0. \quad (2.94)$$

Finally, variations in the Lagrange multipliers ϕ and β_m imply the kinematic equations (2.61) and (2.62), respectively. These two kinematic equations combine with the four variational equations (2.91) through (2.94) to recover the Eulerian motion and micromotion equations after a calculation using the Clebsch relations, the Kelvin-Noether form of the motion equation, and the dynamical equations for the Clebsch potentials.

At the end of in Section 3, we shall systematize this type of calculation and, thus, clarify its meaning as a Poisson map. For now, we simply remark that the evolutionary Clebsch relations are Hamilton's canonical equations for the Hamiltonian obtained from the constrained action in (2.90) by the usual Legendre transformation. Perhaps not unexpectedly, this Hamiltonian agrees exactly with that in equation (2.73) obtained from the Legendre transformation in \mathbf{u} and ν alone.

As we shall discuss more generally in the next section, the Clebsch representations for the momentum and director angular momentum provide a Poisson map from the canonical Poisson bracket in the Clebsch variables to the Lie-Poisson bracket for the Hamiltonian matrix with generalized two-cocycle found in equation (2.82).

Historically, the Hamiltonian approach has been very fruitful in modeling the hydrodynamics of complex fluids and quantum liquids, including superfluids, going back to the seminal work of Khalatnikov and Lebedev [1978], Khalatnikov and Lebedev [1980] and Dzyaloshinskii and Volovick [1980]. The Clebsch approach has provided a series of physical examples of Lie-Poisson brackets: for superfluids in Holm and Kupershmidt [1982]; superconductors in Holm and Kupershmidt [1983a]; Yang-Mills plasmas (chromohydrodynamics) in Gibbons, Holm and Kupershmidt [1982], and Gibbons, Holm and Kupershmidt [1983]; magnetohydrodynamics, multifluid plasmas, and elasticity, in Holm and Kupershmidt [1983b]; Yang-Mills magnetohydrodynamics in Holm and Kupershmidt [1984]; and its relation to superfluid plasmas in Holm and Kupershmidt [1987] and spin-glasses in Holm and Kupershmidt [1988]. Many, but not all, of these Lie-Poisson brackets fit into the present Euler-Poincaré framework for PCFs.

The Euler-Poincaré framework also accomodates many of the various types of Poisson brackets (such as “rigid body fluids”) studied over the years by Grmela, Edwards, Beris, and others, as summarized in Beris and Edwards [1994]. For liquid crystals, these authors develop a bracket description both for Ericksen-Leslie equations and the Doi-Edwards theory based on the conformation tensor \mathbf{C} , which is related to the director theory by $\mathbf{C} = \mathbf{n} \otimes \mathbf{n}$. The extension of the

present results to this case may be accomplished, e.g., by following the Clebsch approach of Holm and Kupershmidt [1983b] who treated the corresponding case of Lie-Poisson brackets for nonlinear elasticity. The treatment in Beris and Edwards [1994] ignores the geometrical content of the Lie-Poisson formulation in preference for its tensor properties alone.

2.4 Summary for perfect liquid crystals

We now recapitulate the steps in the procedure we have followed in deriving the Euler-Lagrange-Poincaré-Clebsch equations and the Lie-Poisson Hamiltonian formulations of the dynamics of perfect liquid crystals.

1. Define the order parameter group and its coset space.
2. Write Hamilton's principle in the Lagrangian fluid description.
3. Make 2 stages of reduction:
 - 1st, to introduce the reduced set of variables in the Lagrangian fluid description; and
 - 2nd, to pass to the Eulerian fluid description.
4. Legendre transform to obtain the Hamiltonian formulation.

The alternative Clebsch procedure starts directly with an action for Hamilton's principle that is defined in the Eulerian fluid description and constrained by the Eulerian kinematic equations. Its Hamiltonian formulation is canonical and passes to a Lie-Poisson formulation via the Poisson map that is defined by the Clebsch representations of the momentum and internal angular momentum in equations (2.91) and (2.92), respectively.

Many physical extensions of these results for perfect liquid crystals are available, e.g., to include MHD, compressibility, anisotropic dielectric and diamagnetic effects, linear wave excitation properties, etc. However, we wish to spend the most of the rest of this paper setting the formulations we have established here for perfect liquid crystal dynamics into the geometrical framework of Lagrangian reduction by stages developed in Cendra, Marsden and Ratiu [1999]. This geometrical setting will take advantage of the unifying interpretation of order parameters as coset spaces of broken symmetry groups. (The coset interpretation of order parameters for liquid crystals, superfluids and spin glasses is reviewed, e.g., in Mermin [1979].) The present formulations are geometrical variants of the Ericksen-Leslie equations for liquid crystal dynamics that illuminate some of their mathematical features from the viewpoint of Lagrangian reduction.

3 Action principles and Lagrangian reduction

As we have seen, the passage to reduced variables $\boldsymbol{\nu}$ and $\boldsymbol{\gamma}_m$ for liquid crystals restricts the variables $\mathbf{n}, \dot{\mathbf{n}}, \nabla \mathbf{n}$ to the coset space $SO(3)/O(2)$ of rotations that properly affect the director \mathbf{n} and imposes invariance of the theory under the reflections $\mathbf{n} \rightarrow -\mathbf{n}$. The reduced variables transform properly under $SO(3)$, because the $O(2)$ isotropy subgroup of \mathbf{n} has been *factored out* of them. Thus, we “mod out” or “reduce” the symmetry-associated degrees of freedom by passing to variables that transform properly under rotations in $SO(3)$ and admit the Z_2 reflections $\mathbf{n} \rightarrow -\mathbf{n}$. The removal of degrees of freedom associated with symmetries is the essential idea behind Marsden-Weinstein group reduction in Marsden and Weinstein [1974]. **Marsden-Weinstein reduction** first appeared

in the Hamiltonian setting. However, this sort of reduction by symmetry groups has been recently extended to the Lagrangian setting, see Cendra, Marsden and Ratiu [1999] and Marsden, Ratiu and Scheurle [1999]. The remainder of the paper applies the mathematical framework of **Lagrangian reduction by stages** due to Cendra, Marsden and Ratiu [1999] to express some of the properties of PCF dynamics in the Eulerian description for an arbitrary order parameter group.

A synthesis of the nonlinear dynamics for the motion and micromotion of various perfect complex fluid models is possible, due to their common mathematical basis. The mathematical basis common to all ideal fluid motion – both classical and complex fluids – is **Hamilton's principle**, see, e.g., Serrin [1959],

$$\delta S = \delta \int L dt = 0. \quad (3.1)$$

In the Lagrangian (or material) representation for fluids, the motion is described by the Euler-Lagrange equations for this action principle.

In the Eulerian (or spatial) representation for fluids, the Euler-Lagrange equations for the dynamics are replaced by the Euler-Poincaré equation. The distinction between Euler-Lagrange equations and Euler-Poincaré equations is exemplified by the distinction between rigid body motion expressed in terms of the Euler angles and their time derivatives on the tangent space $T SO(3)$ of the Lie group of proper rotations $SO(3)$, and that same motion expressed in body angular velocity variables in its Lie algebra $so(3)$. Poincaré [1901] was the first to write the latter equations on an arbitrary Lie algebra; hence, the name **Euler-Poincaré equations**.

Euler-Poincaré equations may be understood and derived via the theory of Lagrangian reduction as in Cendra et al. [1999], and Cendra, Marsden and Ratiu [1999]. Euler-Poincaré equations arise when Euler-Lagrange equations and their corresponding Hamilton principles are mapped from a velocity phase space TQ to the quotient space TQ/G (a vector bundle) by a Lie-group action of a symmetry group G on the configuration space Q . If L is a G -invariant Lagrangian on TQ , this process maps it to a reduced Lagrangian and a corresponding reduced variational principle for the Euler-Poincaré dynamics on TQ/G in which the variations are constrained. See Weinstein [1996] and Cendra, Marsden and Ratiu [1999], for expositions of the mathematical framework that underlies Lagrangian reduction by stages and Holm, Marsden and Ratiu [1998] for a discussion of Euler-Poincaré equations and their many applications in classical ideal fluid dynamics from the viewpoint of the present paper. See Marsden, Ratiu and Scheurle [1999] for additional insight and recent results in Lagrangian reduction.

The order parameters of PCFs are material variables. The Lagrangian in Hamilton's principle (3.1) for PCFs is the map,

$$L : TG \times V^* \times T\mathcal{O} \longmapsto \mathbf{R}. \quad (3.2)$$

That is, the velocity phase space for the PCF Lagrangian L in material variables is the Cartesian product of three spaces:

TG , the tangent space of the Lie group G of fluid motions (the diffeomorphisms that take the fluid parcels from their reference configuration to their current positions in the domain of flow),

V^* , the vector space of advected quantities carried with the fluid motion, and

$T\mathcal{O}$, the tangent space of the Lie group \mathcal{O} of fluid micromotions (\mathcal{O} is the order parameter Lie group).

The advected quantities in V^* include the volume element or mass density and whatever else is carried along with the fluid parcels, such as the magnetic field intensity in the case of magnetohydrodynamics. The new feature of PCFs relative to the simple fluids with advected parameters treated in Holm, Marsden and Ratiu [1998] is the dependence of their Lagrangian on $T\mathcal{O}$. The order parameter coset space at each material point is acted upon by the order parameter Lie group. (We choose the convention of group action from the right.) Since the order parameter is a material property, the *diffeomorphism group G also acts on the order parameter group*, as $\mathcal{O} \times G \rightarrow \mathcal{O}$, denoting action from the right.

In this Section, we shall use Hamilton's principle (3.1) with Lagrangian (3.2) to obtain the dynamical equations for the motion and micromotion of PCFs whose order parameters are defined as coset spaces of Lie groups. In doing so, we shall begin by *assuming* this Lagrangian is invariant under the right action of the order parameter Lie group \mathcal{O} on its tangent space $T\mathcal{O}$. (This right action on the space of internal variables leaves the other components of the configuration space TG and V^* fixed.) We shall assume this Lagrangian is also invariant under the right action of the diffeomorphisms G , which relabel the fluid parcels. (This action of G *does indeed* affect the material variables defined on $T\mathcal{O}$ and V^* .) Under these symmetry assumptions we shall perform the following two group reductions

$$(TG \times V^* \times (T\mathcal{O}/\mathcal{O}))/G \simeq \mathfrak{g} \times (V^* \times \mathfrak{o})g^{-1}(t),$$

with respect to the right actions of first \mathcal{O} and then G , by applying group reduction to the velocity phase space of this Lagrangian in two stages,

$$\text{1st stage, } (TG \times V^* \times T\mathcal{O})/\mathcal{O} \simeq TG \times V^* \times \mathfrak{o}, \quad (3.3)$$

$$\text{2nd stage, } (TG \times V^* \times \mathfrak{o})/G \simeq \mathfrak{g} \times (V^* \times \mathfrak{o})g^{-1}(t). \quad (3.4)$$

Here we denote isomorphisms as, e.g., $\mathfrak{o} \simeq T\mathcal{O}/\mathcal{O}$ and $\mathfrak{g} \simeq TG/G$, where the Lie algebras \mathfrak{o} and \mathfrak{g} correspond, respectively, to the Lie groups \mathcal{O} and G . The first stage is Lagrangian reduction by the right action of \mathcal{O} , the order parameter group.⁵ The second stage is Lagrangian reduction of the first result by the right action of the diffeomorphisms G in the first factor and by composition of functions in the second factor. Because of the assumed invariances of our Lagrangian, these two stages of reduction of the velocity phase spaces will each yield a reduced Lagrangian and a corresponding reduced variational principle for the dynamics. The group actions at each stage are assumed to be free and proper, so the reduced spaces will be local principle fiber bundles.⁶ The mathematical formulation of the process of Lagrangian reduction by stages and the introduction of various connections on the *Lagrange-Poincaré bundles* that arise in Lagrangian reduction are discussed in Cendra, Marsden and Ratiu [1999]. These Lagrange-Poincaré bundles are special cases of *Lie algebroids*. See Weinstein [1996] for a fundamental description of the relation between Lagrangian mechanics and Lie algebroids.

⁵Variants exist. In particular, for liquid crystals, only a part of the Lie algebra \mathfrak{o} is required; namely, $so(3)/O(2)$, the part of the Lie algebra $so(3)$ that is invariant under the $O(2)$ isotropy group of the director, \mathbf{n} .

⁶A natural flat connection appears on this bundle, but this bundle picture should be made intrinsic and global, while including defect dynamics. A strategy for obtaining equations for the defect dynamics will be discussed in the next Section. However, the global bundle picture is for future work.

3.1 Lagrangian reduction by stages

We are dealing with a Lagrangian defined by the map

$$L(g, \dot{g}, a_0, \dot{\chi}, d\chi) : TG \times V^* \times T\mathcal{O} \longmapsto \mathbf{R},$$

where G is the diffeomorphism group that acts on both the vector space V^* of advected material quantities and the order parameter group \mathcal{O} . We assume that L has the following invariance properties,

$$L(g, \dot{g}, a_0, \dot{\chi}, d\chi) = L(g, \dot{g}, a_0, \dot{\chi}\psi, d\chi\psi) = L(gh, \dot{g}h, a_0h, \dot{\chi}\psi h, d\chi\psi h), \quad (3.5)$$

for all $\psi \in \mathcal{O}$ and $h \in G$. In particular, we shall choose $\psi = \chi^{-1}(t)$ in the first stage and $h = g^{-1}(t)$ in the second stage of the reduction, so that

$$L(g, \dot{g}, a_0, \dot{\chi}, d\chi) = L(g, \dot{g}, a_0, \dot{\chi}\chi^{-1}, d\chi\chi^{-1}), \quad (3.6)$$

after the first stage of reduction, and

$$L(g, \dot{g}, a_0, \dot{\chi}, d\chi) = L(e, \dot{g}g^{-1}, a, \dot{\chi}\chi^{-1}g^{-1}, d\chi\chi^{-1}g^{-1}) \equiv l(\xi, a, \nu, \gamma), \quad (3.7)$$

after the second stage, with $\xi \equiv \dot{g}g^{-1}$, $a \equiv a_0g^{-1}$, $\nu \equiv (\dot{\chi}\chi^{-1})g^{-1}$ and $\gamma \cdot dx \equiv (d\chi\chi^{-1})g^{-1}$. After the first stage of reduction, the reduced action principle yields the Lagrange-Poincaré equations, and after the second stage we shall obtain the Euler-Poincaré equations for a perfect complex fluid with an arbitrary order parameter group.

3.1.1 Lagrange-Poincaré equations

The first stage

$$TG \times V^* \times T\mathcal{O} \longmapsto TG \times V^* \times \mathfrak{o},$$

of the two-stage symmetry reduction in (3.3) - (3.4) affects only the internal variables and passes from coordinates on the order parameter Lie group, \mathcal{O} , to coordinates on its Lie algebra, \mathfrak{o} , obtained from the tangent vectors of the order parameter Lie group at the identity by the isomorphism $\mathfrak{o} \simeq T\mathcal{O}/\mathcal{O}$. The results at this first stage consist of Euler-Lagrange equations for the fluid motion, coupled through additional components of the stress tensor to equations of a type called Lagrange-Poincaré equations in Marsden and Scheurle [1995], Cendra, Marsden and Ratiu [1999]. In our case, these Lagrange-Poincaré equations describe the micromotion in the Lagrangian (or material) fluid description.

The first stage of reduction results in the *Lagrange-Poincaré action principle*,

$$\delta \int dt \int d^3X L(\dot{x}, J, \nu, \gamma) = 0, \quad (3.8)$$

written in the material representation and denoted as follows,

$L(\dot{x}, J, \nu, \gamma)$ is the reduced Lagrangian on $TG \times V^* \times \mathfrak{o}$,

$J(X, t) = \det(\partial x / \partial X) \in V^*$ is the volume element,

$\nu(X, t) = \dot{\chi}\chi^{-1}(X, t) \in \mathfrak{o}$ is the material angular frequency and

$\gamma \cdot dx = d\chi\chi^{-1}(X, t) \in \mathfrak{o}$ with components denoted as γ_m given by

$$\begin{aligned} \gamma_m dx_m(X, t) &= d\chi\chi^{-1}(X, t), \\ &= \gamma_m \frac{\partial x_m}{\partial X_A} dX_A = \gamma_A^{mat}(X, t) dX_A, \end{aligned} \quad (3.9)$$

where $\gamma \cdot dx$ is the Cosserat strain one-form introduced in Cosserat [1909] and superposed “dot” $(\cdot)^\bullet$ denotes time derivative at *fixed material position* X .

These material quantities satisfy auxiliary **kinematic equations**, obtained by differentiating their definitions,

$$(J^{-1}d^3x)^\bullet = (d^3X)^\bullet = 0, \quad (3.10)$$

$$(\gamma \cdot dx)^\bullet = (d\chi\chi^{-1})^\bullet = d\nu + \text{ad}_\nu(\gamma \cdot dx), \quad (3.11)$$

The material angular frequency $\nu = \dot{\chi}\chi^{-1}$ and the material Cosserat strain one-form $\gamma \cdot dx = d\chi\chi^{-1}$ take their values in the right-invariant Lie algebra \mathfrak{o} of the order parameter Lie group \mathcal{O} . The **ad-operation** appearing in equation (3.11) denotes multiplication, or commutator, in the Lie algebra \mathfrak{o} .

The dynamical **Lagrange-Poincaré equations** determine the complex fluid’s motion with fluid trajectory $\phi_t(X) = x(X, t)$ with $\phi_t \in G$ and micromotion $\chi(X, t) \in \mathcal{O}$ in the material fluid description. These equations take the following forms,

$$J^{-1} \left(\frac{\partial L}{\partial \dot{x}_p} \right)^\bullet + \frac{\partial}{\partial x_p} \frac{\partial L}{\partial J} - \frac{\partial}{\partial x_m} \left\langle J^{-1} \frac{\partial L}{\partial \gamma_m}, \gamma_p \right\rangle = 0, \quad (3.12)$$

$$\left(\frac{\partial L}{\partial \nu} \right)^\bullet - \text{ad}_\nu^* \frac{\partial L}{\partial \nu} + J \frac{\partial}{\partial x_m} \left(J^{-1} \frac{\partial L}{\partial \gamma_m} \right) - \text{ad}_{\gamma_m}^* \frac{\partial L}{\partial \gamma_m} = 0. \quad (3.13)$$

The **ad*-operation** appearing in (3.13) is defined in terms of the ad-operation and the symmetric pairing $\langle \cdot, \cdot \rangle : \mathfrak{o}^* \times \mathfrak{o} \rightarrow \mathbf{R}$ between elements of the right Lie algebra \mathfrak{o} and its dual \mathfrak{o}^* as, e.g.,

$$-\left\langle \text{ad}_\nu^* \frac{\partial L}{\partial \nu}, \Sigma \right\rangle = \left\langle \frac{\partial L}{\partial \nu}, \text{ad}_\nu \Sigma \right\rangle = \left\langle \frac{\partial L}{\partial \nu}, [\nu, \Sigma] \right\rangle. \quad (3.14)$$

In a Lie algebra basis satisfying $[e_\alpha, e_\beta] = t_{\alpha\beta}^\iota e_\iota$ and its dual basis e^κ satisfying $\langle e^\kappa, e_\iota \rangle = \delta_\iota^\kappa$, we may write this formula as

$$-\left\langle \text{ad}_\nu^* \frac{\partial L}{\partial \nu}, \Sigma \right\rangle = \left\langle \frac{\partial L}{\partial \nu}, \text{ad}_\nu \Sigma \right\rangle = \frac{\partial L}{\partial \nu^\kappa} t_{\alpha\beta}^\kappa \nu^\alpha \Sigma^\beta. \quad (3.15)$$

Thus, for the sign conventions we choose in (3.14), the ad*-operation is defined as the **negative transpose** of the ad-operation.

The dynamical equations (3.12) and (3.13) follow from the Lagrange-Poincaré action principle (3.8) for PCF dynamics in the material fluid description. These Lagrange-Poincaré equations may

be calculated directly, as

$$\begin{aligned}
0 &= \delta \int dt \int d^3 X L(\dot{x}, J, \nu, \gamma) \\
&= \int dt \int d^3 X \left[\frac{\partial L}{\partial \dot{x}_p} \delta \dot{x}_p + \frac{\partial L}{\partial J} \delta J + \left\langle \frac{\partial L}{\partial \nu}, \delta \nu \right\rangle + \left\langle \frac{\partial L}{\partial \gamma_m}, \delta \gamma_m \right\rangle \right] \\
&= \int dt \int d^3 X \left\{ \delta x_p \left[- \left(\frac{\partial L}{\partial \dot{x}_p} \right)^\cdot - J \frac{\partial}{\partial x_p} \frac{\partial L}{\partial J} + J \frac{\partial}{\partial x_m} \left\langle J^{-1} \frac{\partial L}{\partial \gamma_m}, \gamma_p \right\rangle \right] \right. \\
&\quad \left. + \left\langle \left[- \left(\frac{\partial L}{\partial \nu} \right)^\cdot + \text{ad}_\nu^* \frac{\partial L}{\partial \nu} - J \frac{\partial}{\partial x_m} \left(J^{-1} \frac{\partial L}{\partial \gamma_m} \right) + \text{ad}_{\gamma_m}^* \frac{\partial L}{\partial \gamma_m} \right], \Sigma \right\rangle \right\} \\
&\quad + \int dt \oint d^2 S \hat{n}_A \frac{\partial X_A}{\partial x_m} \left(\left\langle \frac{\partial L}{\partial \gamma_m}, \Sigma \right\rangle - \left\langle \frac{\partial L}{\partial \gamma_m}, \gamma_p \right\rangle \delta x_p + J \frac{\partial L}{\partial J} \delta x_m \right).
\end{aligned} \tag{3.16}$$

Here we define $\Sigma \equiv \delta \chi \chi^{-1}$, in terms of which we calculate,

$$\delta \nu = \dot{\Sigma} - [\nu, \Sigma] = \dot{\Sigma} - \text{ad}_\nu \Sigma, \tag{3.17}$$

$$\delta \gamma_A^{mat} = \frac{\partial \Sigma}{\partial X_A} - [\gamma_A^{mat}, \Sigma] = \frac{\partial \Sigma}{\partial X_A} - \text{ad}_{\gamma_A^{mat}} \Sigma. \tag{3.18}$$

Since $\gamma_m = \gamma_A^{mat}(\partial X_A / \partial x_m)$, this means that

$$\begin{aligned}
\delta \gamma_m &= -\gamma_p \frac{\partial X_B}{\partial x_m} \frac{\partial}{\partial X_B} \delta x_p + \frac{\partial X_A}{\partial x_m} \delta \gamma_A^{mat} \\
&= -\gamma_p \frac{\partial X_B}{\partial x_m} \frac{\partial}{\partial X_B} \delta x_p + \frac{\partial X_A}{\partial x_m} \frac{\partial \Sigma}{\partial X_A} - \text{ad}_{\gamma_m} \Sigma.
\end{aligned} \tag{3.19}$$

Thus, the variations in γ_m couple the two Lagrange-Poincaré equations. We also drop endpoint terms that arise from integrating by parts in time, upon taking δx_p and Σ to vanish at these endpoints. The natural boundary conditions

$$\frac{\partial \mathcal{L}}{\partial J} = 0 \quad \text{and} \quad \hat{n}_m \frac{\partial \mathcal{L}}{\partial \gamma_m} = 0, \tag{3.20}$$

ensure that the fluid pressure and the normal stress are continuous across a fluid interface.

3.1.2 Euler-Poincaré equations

The passage next from the Lagrangian fluid description of continuum mechanics to the Eulerian fluid description will yield the Euler-Poincaré equations. We obtain these equations by applying to Hamilton's principle the second stage,

$$TG \times V^* \times \mathfrak{o} \longmapsto \mathfrak{g} \times (V^* \times \mathfrak{o}) g^{-1}(t),$$

of the two-stage Lagrangian reduction in (3.3) - (3.4). This second stage of reduction results in the ***Euler-Poincaré action principle***,

$$\delta \int l(\xi, a, \nu, \gamma) dt = 0, \tag{3.21}$$

with constrained variations

$$\begin{aligned}\delta\xi &= \frac{\partial\eta}{\partial t} + [\xi, \eta], \quad \delta a = -a\eta, \\ \delta\nu &= \frac{\partial\Sigma}{\partial t} + \xi \cdot \nabla \Sigma - \text{ad}_\nu \Sigma - \nu\eta, \\ \delta(\gamma \cdot dx) &= d\Sigma + \text{ad}_\Sigma(\gamma \cdot dx) - (\gamma \cdot dx)\eta,\end{aligned}\tag{3.22}$$

where $\eta(t) = \delta g(t)g(t)^{-1} \in \mathfrak{g}$ and $\Sigma(t) = \delta\chi(t)\chi(t)^{-1} \in \mathfrak{o}$ both vanish at the endpoints.

The Euler-Poincaré action principle produces the following equations defined on $\mathfrak{g} \times (V^* \times \mathfrak{o})g^{-1}(t)$ for the motion and micromotion, in which $\partial/\partial t$ denotes Eulerian time derivative at *fixed spatial position* x ,

$$\frac{\partial}{\partial t} \frac{\delta l}{\delta \xi} = -\text{ad}_\xi^* \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a + \frac{\delta l}{\delta \nu} \diamond \nu + \frac{\delta l}{\delta \gamma_m} \diamond \gamma_m,\tag{3.23}$$

$$\frac{\partial}{\partial t} \frac{\delta l}{\delta \nu} = -\text{div}\left(\xi \frac{\delta l}{\delta \nu} + \frac{\delta l}{\delta \gamma}\right) + \text{ad}_\nu^* \frac{\delta l}{\delta \nu} + \text{ad}_{\gamma_m}^* \frac{\delta l}{\delta \gamma_m}.\tag{3.24}$$

These are the ***Euler-Poincaré equations*** for a perfect complex fluid. In these equations, l is the reduced Lagrangian on $\mathfrak{g} \times (V^* \times \mathfrak{o})g^{-1}(t)$. Also,

$$\text{ad}_\xi \eta \equiv [\xi, \eta], \quad \text{with } \xi, \eta \in \mathfrak{g},\tag{3.25}$$

is the commutator in the Lie algebra of vector fields, \mathfrak{g} . In addition, we define the two operations ad_ξ^* and \diamond as

$$\left\langle \text{ad}_\xi^* \frac{\delta l}{\delta \xi}, \eta \right\rangle \equiv - \left\langle \frac{\delta l}{\delta \xi}, \text{ad}_\xi \eta \right\rangle,\tag{3.26}$$

and

$$\left\langle \frac{\delta l}{\delta a} \diamond a, \eta \right\rangle \equiv - \left\langle \frac{\delta l}{\delta a}, a\eta \right\rangle.\tag{3.27}$$

The concatenation $a\eta$ denotes the right Lie algebra action of $\eta \in \mathfrak{g}$ on $a \in V^*$ (by Lie derivative). The pairing $\langle \cdot, \cdot \rangle$ now includes *spatial integration* and, thus, allows for integration by parts. Similar definitions hold for $(\delta l/\delta \gamma_m \diamond \gamma_m)$ and $(\delta l/\delta \nu \diamond \nu)$.

In components, these quantities are given by,

$$\begin{aligned}\left\langle \frac{\delta l}{\delta \gamma_m} \diamond \gamma_m, \eta \right\rangle &= - \left\langle \frac{\delta l}{\delta \gamma_m}, \gamma_m \eta \right\rangle = \left\langle \frac{\partial}{\partial x_m} \left(\frac{\delta l}{\delta \gamma_m^\beta} \gamma_j^\beta \right) - \frac{\delta l}{\delta \gamma_m^\beta} \frac{\partial \gamma_m^\beta}{\partial x_j}, \eta_j \right\rangle, \\ \left\langle \frac{\delta l}{\delta \nu} \diamond \nu, \eta \right\rangle &= - \left\langle \frac{\delta l}{\delta \nu}, \nu \eta \right\rangle = \left\langle - \frac{\delta l}{\delta \nu^\beta} \frac{\partial \nu^\beta}{\partial x_j}, \eta_j \right\rangle, \\ \left\langle \frac{\delta l}{\delta D} \diamond D, \eta \right\rangle &= - \left\langle \frac{\delta l}{\delta D}, D \eta \right\rangle = \left\langle D \frac{\partial}{\partial x_j} \frac{\delta l}{\delta D}, \eta_j \right\rangle.\end{aligned}\tag{3.28}$$

Remark. At this point, one might have also introduced $(3 + 1)$ *covariant derivatives* acting on \mathfrak{o} -valued functions of space and time, as

$$D_m \equiv \frac{\partial}{\partial x_m} - \text{ad}_{\gamma_m}, \quad \text{and} \quad D_t \equiv \frac{\partial}{\partial t} - \text{ad}_\nu, \quad (3.29)$$

with associated curvature (or Yang-Mills magnetic field) given by

$$\text{ad}_{B_{ij}} \equiv [D_i, D_j], \quad (3.30)$$

whose components are expressed, cf. equation (2.28),

$$B_{ij}^\alpha = \gamma_{i,j}^\alpha - \gamma_{j,i}^\alpha + t_{\beta\kappa}^\alpha \gamma_i^\beta \gamma_j^\kappa, \quad (3.31)$$

as in Holm and Kupershmidt [1988]. However, the operations ad , ad^* , \diamond and Lie derivative are sufficient for our present purposes.

Eulerian kinematic equations. By definition, an *Eulerian advected quantity* $a \in V^*g(t)^{-1}$ satisfies

$$\frac{\partial a}{\partial t} + a \xi = 0.$$

This advection relation may be written equivalently as

$$\frac{\partial a}{\partial t} + \mathcal{L}_\xi a = 0,$$

where \mathcal{L}_ξ is the Lie derivative with respect to $\xi = \dot{g}(t)g^{-1}(t)$, the Eulerian fluid velocity, often denoted also as $u(x, t)$.

The Eulerian versions of the Lagrangian *kinematic equations* (3.10) and (3.11) are given in terms of the Lie derivative by

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_\xi \right) (D d^3 x) = 0, \quad (3.32)$$

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_\xi \right) (\gamma \cdot dx) = d\nu + \text{ad}_\nu(\gamma \cdot dx). \quad (3.33)$$

In these equations, the quantity $D(x, t) = J^{-1}(X, t)g(t)^{-1}$ is the Eulerian mass density and the quantities $\nu(x, t) = \nu(X, t)g(t)^{-1}$ and

$$\gamma(x, t) \cdot dx = \left(\gamma(X, t) \cdot \frac{\partial x}{\partial X}(X, t) \cdot dX \right) g(t)^{-1}$$

are the Eulerian counterparts of the right-invariant material quantities $\nu(X, t)$ and $\gamma(X, t)$ in equations (3.11).

Remark. We note that equation (3.33) implies the γ -**circulation theorem** for PCFs, cf. equation (2.63),

$$\frac{d}{dt} \oint_{c(\xi)} \gamma \cdot dx = \oint_{c(\xi)} \text{ad}_\nu(\gamma \cdot dx). \quad (3.34)$$

Thus, the circulation of γ around a loop $c(\xi)$ moving with the fluid is conserved when $\text{ad}_\nu \gamma$ is a gradient. Otherwise, the curl of this quantity generates circulation of γ around fluid loops.

The Euler-Poincaré equations (3.23) and (3.24) may be obtained directly from the Euler-Poincaré action principle (3.21), as follows.

Euler-Poincaré action variations. We compute the variation of the action (3.21) in Eulerian variables at fixed time t and spatial position \mathbf{x} as,

$$\begin{aligned} \delta \mathcal{S} &= \int dt \left[\left\langle \frac{\delta l}{\delta \xi}, \delta \xi \right\rangle + \left\langle \frac{\delta l}{\delta a}, \delta a \right\rangle + \left\langle \frac{\delta l}{\delta \nu}, \delta \nu \right\rangle + \left\langle \frac{\delta l}{\delta \gamma_m}, \delta \gamma_m \right\rangle \right] \\ &= \int dt \left\{ \left\langle -\frac{\partial}{\partial t} \frac{\delta l}{\delta \xi} - \text{ad}_\xi^* \frac{\delta l}{\delta \xi} + a \diamond \frac{\delta l}{\delta a} + \nu \diamond \frac{\delta l}{\delta \nu} + \gamma_m \diamond \frac{\delta l}{\delta \gamma_m}, \eta \right\rangle \right. \\ &\quad + \left\langle -\frac{\partial}{\partial t} \frac{\delta l}{\delta \nu} - \frac{\partial}{\partial x_m} \left(\frac{\delta l}{\delta \nu} \xi_m + \frac{\delta l}{\delta \gamma_m} \right) + \text{ad}_\nu^* \frac{\delta l}{\delta \nu} + \text{ad}_{\gamma_m}^* \frac{\delta l}{\delta \gamma_m}, \Sigma \right\rangle \left. \right\} \\ &\quad + \int dt \int d^3 x \left\{ \frac{\partial}{\partial t} \left[\frac{\delta l}{\delta \xi_j} \eta_j + \frac{\delta l}{\delta \nu^\beta} \Sigma^\beta \right] - d \left(\eta \rfloor a, \frac{\delta l}{\delta a} \right) \right. \\ &\quad \left. + \frac{\partial}{\partial x_m} \left[\eta_j \left(\xi_m \frac{\delta l}{\delta \xi_j} - \frac{\delta l}{\delta \gamma_m^\beta} \gamma_j^\beta \right) + \left(\frac{\delta l}{\delta \nu^\beta} \xi_m + \frac{\delta l}{\delta \gamma_m^\beta} \right) \Sigma^\beta \right] \right\}, \end{aligned} \quad (3.35)$$

where we have used the variational expressions in (3.22) and integrated by parts. Here $\eta \rfloor a$ in the first boundary term denotes substitution of the vector field η into the tensor differential form a and (\cdot, \cdot) denotes the natural pairing between a tensor field and its dual.

The dynamical Euler-Poincaré equations (3.23) and (3.24) are thus obtained from the Euler-Poincaré action principle (3.21), by requiring the coefficients of the arbitrary variations η and Σ to vanish in the variational formula (3.35). The remaining terms in (3.35) yield **Noether's theorem** for this system, which assigns a conservation law to each symmetry of the Euler-Poincaré variational principle.

Momentum conservation. In momentum conservation form, the PCF motion equation in the Eulerian fluid description (3.23) becomes, for *algebraic* dependence of the Lagrangian density l on (ξ, a, ν, γ_m) ,

$$\frac{\partial}{\partial t} \frac{\partial l}{\partial \xi_j} = - \frac{\partial}{\partial x_m} \left(\xi_m \frac{\partial l}{\partial \xi_j} + l \delta_{mj} - \frac{\partial l}{\partial \gamma_m^\beta} \gamma_j^\beta \right) + d \left(\frac{\partial}{\partial x_j} \rfloor a, \frac{\partial l}{\partial a} \right). \quad (3.36)$$

In this equation, expressed in Cartesian coordinates, there is an implied sum over the various types of advected tensor quantities, a . This momentum conservation law also arises from Noether's

theorem, as a consequence of the invariance of the variational principle (3.21) under spatial translations. In fact, the simplest derivation of this equation is obtained by evaluating the variational formula (3.35) on the equations of motion and using the translational symmetry of the Lagrangian in Noether's theorem with $\eta_j = \partial/\partial x_j$. For an algebraic Lagrangian density $l(\xi, D, \nu, \gamma_m)$, this momentum conservation law becomes, cf. equation (2.68),

$$\frac{\partial}{\partial t} \frac{\partial l}{\partial \xi_j} = - \frac{\partial}{\partial x_m} \left(\xi_m \frac{\partial l}{\partial \xi_j} + \left(l - D \frac{\partial l}{\partial D} \right) \delta_{mj} - \frac{\partial l}{\partial \gamma_m^\beta} \gamma_j^\beta \right). \quad (3.37)$$

Kelvin-Noether circulation theorem for PCFs. Rearranging the motion equation (3.23) and using the continuity equation for D in (3.32) gives

$$\frac{d}{dt} \oint_{c(\xi)} \frac{1}{D} \frac{\delta l}{\delta \xi_j} dx_j = \oint_{c(\xi)} \frac{1}{D} \left[\frac{\delta l}{\delta a} \diamond a + \frac{\delta l}{\delta \nu} \diamond \nu + \frac{\delta l}{\delta \gamma_m} \diamond \gamma_m \right], \quad (3.38)$$

where the circulation loop $c(\xi)$ moves with the fluid velocity ξ and we have used the following relation, valid for one-form densities,

$$\text{ad}_\xi^* \frac{\delta l}{\delta \xi} = \mathcal{L}_\xi \frac{\delta l}{\delta \xi}, \quad \text{with} \quad \frac{\delta l}{\delta \xi} = \left(\frac{\delta l}{\delta \xi_j} dx_j \otimes d^3x \right). \quad (3.39)$$

This relation may be checked explicitly in Cartesian coordinates, as follows,

$$\begin{aligned} \left\langle \mathcal{L}_\xi \frac{\delta l}{\delta \xi}, \eta \right\rangle &= \int \left(\frac{\partial}{\partial x^j} \left(\xi^j \frac{\delta l}{\delta \xi^i} \right) + \frac{\delta l}{\delta \xi^j} \frac{\partial \xi^j}{\partial x^i} \right) \eta^i d^3x \\ &= - \int \frac{\delta l}{\delta \xi^j} \left(\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i} \right) d^3x \\ &= - \left\langle \frac{\delta l}{\delta \xi}, \text{ad}_\xi \eta \right\rangle = \left\langle \text{ad}_\xi^* \frac{\delta l}{\delta \xi}, \eta \right\rangle. \end{aligned} \quad (3.40)$$

See Holm, Marsden and Ratiu [1998] for more explanation and discussion of the Kelvin-Noether circulation theorem for Euler-Poincaré systems.

In components, the Kelvin-Noether circulation theorem (3.38) for PCFs with Lagrangian $l(\xi, D, \nu, \gamma_m)$ may be written using equation (3.28) as,

$$\frac{d}{dt} \oint_{c(\xi)} \frac{1}{D} \frac{\delta l}{\delta \xi_j} dx_j = \oint_{c(\xi)} \frac{1}{D} \left[D \frac{\partial}{\partial x_j} \frac{\delta l}{\delta D} - \frac{\delta l}{\delta \nu^\beta} \frac{\partial \nu^\beta}{\partial x_j} + \frac{\partial}{\partial x_m} \left(\frac{\delta l}{\delta \gamma_m^\beta} \gamma_j^\beta \right) - \frac{\delta l}{\delta \gamma_m^\beta} \frac{\partial \gamma_m^\beta}{\partial x_j} \right] dx_j. \quad (3.41)$$

Thus, gradients of angular frequency and order parameter strain may cause fluid circulation.

3.2 Hamiltonian dynamics of PCFs

The Legendre Transformation. One passes from Euler-Poincaré equations on a Lie algebra \mathfrak{g} to Lie-Poisson equations on the dual \mathfrak{g}^* by means of the Legendre transformation, see, e.g., Holm, Marsden and Ratiu [1998]. In our case, we start with the reduced Lagrangian l on $\mathfrak{g} \times (V^* \times \mathfrak{o})g(t)^{-1}$ and perform a Legendre transformation in the variables ξ and ν only, by writing

$$\mu = \frac{\delta l}{\delta \xi}, \quad \sigma = \frac{\delta l}{\delta \nu}, \quad h(\mu, a, \sigma, \gamma) = \langle \mu, \xi \rangle + \langle \sigma, \nu \rangle - l(\xi, a, \nu, \gamma). \quad (3.42)$$

One then computes the variational derivatives of h as

$$\frac{\delta h}{\delta \mu} = \xi, \quad \frac{\delta h}{\delta \sigma} = \nu, \quad \frac{\delta h}{\delta a} = -\frac{\delta l}{\delta a}, \quad \frac{\delta h}{\delta \gamma} = -\frac{\delta l}{\delta \gamma}. \quad (3.43)$$

Consequently, the Euler-Poincaré equations (3.23) - (3.33) for PCF dynamics in the Eulerian description imply the following equations, for the Legendre-transformed variables, (μ, a, σ, γ) , cf. equations (2.75) - (2.78)

$$\begin{aligned} \frac{\partial \mu}{\partial t} &= -\text{ad}_{\delta h / \delta \mu}^* \mu - \frac{\delta h}{\delta a} \diamond a - \frac{\delta h}{\delta \gamma_m} \diamond \gamma_m - \frac{\delta h}{\delta \sigma} \diamond \sigma, \\ \frac{\partial a}{\partial t} &= -\mathcal{L}_{\delta h / \delta \mu} a, \\ \frac{\partial \gamma}{\partial t} \cdot dx &= -\mathcal{L}_{\delta h / \delta \mu} (\gamma \cdot dx) + d\left(\frac{\delta h}{\delta \sigma}\right) + \text{ad}_{\delta h / \delta \sigma} (\gamma \cdot dx), \\ \frac{\partial \sigma}{\partial t} &= -\text{div} \left(\frac{\delta h}{\delta \mu} \sigma - \frac{\delta h}{\delta \gamma} \right) + \text{ad}_{\delta h / \delta \sigma}^* \sigma - \text{ad}_{\gamma_m}^* \frac{\delta h}{\delta \gamma_m}. \end{aligned} \quad (3.44)$$

As for the case of liquid crystals discussed earlier, these equations are **Hamiltonian** and may be expressed in terms of a Lie-Poisson bracket.

Lie-Poisson bracket for PCFs. Assembling the PCF equations (3.44) into Hamiltonian form gives, symbolically,

$$\frac{\partial}{\partial t} \begin{bmatrix} \mu \\ a \\ \gamma \\ \sigma \end{bmatrix} = - \begin{bmatrix} \text{ad}_{\square}^* \mu & \square \diamond a & \square \diamond \gamma & \square \diamond \sigma \\ \mathcal{L}_{\square} a & 0 & 0 & 0 \\ \mathcal{L}_{\square} \gamma & 0 & 0 & -(\text{grad} - \text{ad}_{\gamma}) \square \\ \mathcal{L}_{\square} \sigma & 0 & -(\text{div} - \text{ad}_{\gamma}^*) \square & -\text{ad}_{\square}^* \sigma \end{bmatrix} \begin{bmatrix} \delta h / \delta \mu \\ \delta h / \delta a \\ \delta h / \delta \gamma \\ \delta h / \delta \sigma \end{bmatrix} \quad (3.45)$$

with boxes \square indicating where the matrix operations occur. In the $\gamma - \sigma$ entry of the Hamiltonian matrix (3.45), one recognizes the **covariant spatial derivative** defined in equation (3.29), and finds its adjoint operator in the $\sigma - \gamma$ entry. More explicitly, in terms of indices and differential operators, and for $a = D$, the mass density, this Hamiltonian matrix form becomes

$$\frac{\partial}{\partial t} \begin{bmatrix} \mu_i \\ D \\ \gamma_i^\alpha \\ \sigma_\alpha \end{bmatrix} = - \begin{bmatrix} \mu_j \partial_i + \partial_j \mu_i & D \partial_i & \partial_j \gamma_i^\beta - \gamma_{j,i}^\beta & \sigma_\beta \partial_i \\ \partial_j D & 0 & 0 & 0 \\ \gamma_j^\alpha \partial_i + \gamma_{i,j}^\alpha & 0 & 0 & -\delta_\beta^\alpha \partial_i - t_{\beta\kappa}^\alpha \gamma_i^\kappa \\ \partial_j \sigma_\alpha & 0 & -\delta_\alpha^\beta \partial_j + t_{\alpha\kappa}^\beta \gamma_j^\kappa & -t_{\alpha\beta}^\kappa \sigma_\kappa \end{bmatrix} \begin{bmatrix} \delta h / \delta \mu_j \\ \delta h / \delta D \\ \delta h / \delta \gamma_j^\beta \\ \delta h / \delta \sigma_\beta \end{bmatrix} \quad (3.46)$$

Here, the summation convention is enforced on repeated indices. Upper Greek indices refer to the Lie algebraic basis set, lower Greek indices refer to the dual basis and Latin indices refer to the spatial reference frame. The partial derivative $\partial_j = \partial / \partial x_j$, say, acts to the right on all terms in a product by the chain rule. For the case that $t_{\beta\kappa}^\alpha$ are structure constants $\epsilon_{\alpha\beta\kappa}$ for the Lie algebra $so(3)$, the Lie-Poisson Hamiltonian matrix (2.82) for liquid crystals is recovered, modulo an inessential factor of 2.

Remark. As mentioned earlier in our discussion of Hamiltonian dynamics of liquid crystals, the Hamiltonian matrix in equation (3.46) was discovered some time ago in the context of investigating the relation between spin-glasses and Yang-Mills magnetohydrodynamics (YM-MHD) by using

the Hamiltonian approach in Holm and Kupershmidt [1988]. There, it was shown to be a valid Hamiltonian matrix by associating its Poisson bracket with the dual space of a certain Lie algebra of semidirect-product type that has a **generalized two-cocycle** on it. This generalized two-cocycle contributes the grad and div terms appearing in the more symbolic expression of this Hamiltonian matrix in equation (3.45).

The mathematical discussion of this Lie algebra and its generalized two-cocycle, as well as the corresponding Lie-Poisson Hamiltonian equations for spin-glass fluids and YM-MHD, are given in Holm and Kupershmidt [1988]. The present work provides a rationale for the derivation of such Lie-Poisson brackets from the Lagrangian side.

Spatially one-dimensional static solutions with z -variation. Static (steady, zero-velocity) solutions for PCFs, with constant pressure and one-dimensional spatial variations in, say, the z -direction obey equations (3.46) for $i = 3 = j$, rewritten as

$$\begin{aligned} -D \frac{d}{dz} \frac{\delta h}{\delta D} &= \gamma_3^\alpha \frac{d}{dz} \frac{\delta h}{\delta \gamma_3^\alpha} + \sigma_\alpha \frac{d}{dz} \frac{\delta h}{\delta \sigma_\alpha} = 0, \\ \frac{d}{dz} \frac{\delta h}{\delta \gamma_3^\alpha} &= t_{\alpha\kappa}^\beta \gamma_3^\kappa \frac{\delta h}{\delta \gamma_3^\beta} - t_{\alpha\beta}^\kappa \sigma_\kappa \frac{\delta h}{\delta \sigma_\beta}, \\ \frac{d}{dz} \frac{\delta h}{\delta \sigma_\alpha} &= -t_{\beta\kappa}^\alpha \gamma_3^\kappa \frac{\delta h}{\delta \sigma_\beta}. \end{aligned} \quad (3.47)$$

As for the case of liquid crystals, the sum of terms in the first equation of the set (3.47) vanishes to give zero pressure gradient, as a consequence of the latter two equations.

Under the Legendre transformation

$$h(\gamma_3, \sigma) = \Pi_\alpha \gamma_3^\alpha + \Gamma^\alpha \sigma_\alpha - H(\Pi, \Gamma), \quad (3.48)$$

these equations become,

$$\begin{aligned} \frac{d}{dz} \Pi_\alpha &= t_{\alpha\kappa}^\beta \frac{\delta H}{\delta \Pi_\kappa} \Pi_\beta - t_{\alpha\beta}^\kappa \frac{\delta H}{\delta \Gamma^\kappa} \Gamma^\beta, \\ \frac{d}{dz} \Gamma^\alpha &= -t_{\beta\kappa}^\alpha \frac{\delta H}{\delta \Pi_\kappa} \Gamma^\beta. \end{aligned} \quad (3.49)$$

These Legendre-transformed equations are Poincaré's [1901] generalization of Euler's equations for a heavy top, expressing them on an arbitrary Lie algebra with structure constants $t_{\beta\kappa}^\alpha$. Thus,

The steady, spatially one-dimensional solutions for all PCFs have the underlying Lie algebra structure discovered in Poincaré [1901].

Spatially homogeneous, time-dependent PCF flows. Spatially homogeneous solutions of equations (3.46) obey the dynamical equations,

$$\begin{aligned} \frac{d\sigma_\alpha}{dt} &= -t_{\alpha\kappa}^\beta \gamma_j^\kappa \frac{\delta h}{\delta \gamma_j^\beta} + t_{\alpha\beta}^\kappa \sigma_\kappa \frac{\delta h}{\delta \sigma_\beta}, \\ \frac{d\gamma_i^\alpha}{dt} &= t_{\beta\kappa}^\alpha \gamma_i^\kappa \frac{\delta h}{\delta \sigma_\beta}. \end{aligned} \quad (3.50)$$

For a single spatial index, say γ_i , these are again Poincaré's [1901] equations generalizing Euler's equations for a heavy top to an arbitrary Lie algebra. Of course, the corresponding Hamiltonian matrix for this system is the lower right corner of the matrix in equation (3.46).

When $t_{\beta\kappa}^\alpha = \epsilon_{\alpha\beta\kappa}$ for the Lie algebra $so(3)$, Poincaré's equations (3.50) correspond to the **Leggett equations** for ${}^3He - A$ with spin density σ_α and spin anisotropy vector γ_i^α , see Leggett [1975]. For special solutions of these and other related equations in the context of ${}^3He - A$, see Golo and Monastyrskii [1977], Golo and Monastyrskii [1978], Golo et al. [1979].

Evolution of the disclination density. Holm and Kupershmidt [1988] use the chain rule and the defining relation for the disclination density, cf. equation (2.28),

$$B_{ij}^\alpha \equiv \gamma_{i,j}^\alpha - \gamma_{j,i}^\alpha + t_{\beta\kappa}^\alpha \gamma_i^\beta \gamma_j^\kappa, \quad (3.51)$$

to transform the Hamiltonian matrix (3.46) to a new Hamiltonian matrix, whose Lie-Poisson Hamiltonian dynamics may be written as

$$\begin{aligned} & \frac{\partial}{\partial t} \begin{bmatrix} \mu_i \\ D \\ B_{ij}^\alpha \\ \sigma_\alpha \end{bmatrix} \\ &= - \begin{bmatrix} \mu_k \partial_i + \partial_k \mu_i & D \partial_i & -B_{lm,i}^\beta + \partial_m B_{li}^\beta - \partial_l B_{mi}^\beta & \sigma_\beta \partial_i \\ \partial_k D & 0 & 0 & 0 \\ B_{ij,k}^\alpha + B_{ik}^\alpha \partial_j - B_{jk}^\alpha \partial_i & 0 & 0 & -t_{\beta\kappa}^\alpha B_{ij}^\kappa \\ \partial_k \sigma_\alpha & 0 & t_{\alpha\kappa}^\beta B_{lm}^\kappa & -t_{\alpha\beta}^\kappa \sigma_\kappa \end{bmatrix} \begin{bmatrix} \delta h / \delta \mu_k \\ \delta h / \delta D \\ \delta h / \delta B_{lm}^\beta \\ \delta h / \delta \sigma_\beta \end{bmatrix} \end{aligned} \quad (3.52)$$

The corresponding **PCF dynamics for the disclination density** emerges as

$$\frac{\partial B_{ij}^\alpha}{\partial t} = - \left(B_{ij,k}^\alpha + B_{ik}^\alpha \partial_j - B_{jk}^\alpha \partial_i \right) \frac{\delta h}{\delta \mu_k} + t_{\beta\kappa}^\alpha B_{ij}^\kappa \frac{\delta h}{\delta \sigma_\beta}. \quad (3.53)$$

As expected, this equation preserves the trivial solution $B_{ij}^\alpha = 0$, which is the case when the γ_m -strain field is continuous and the complex fluid has no defects. We shall mention a strategy for dealing with defects in Section 4.

3.3 Clebsch approach for PCF dynamics

Following Serrin [1959], we call the auxiliary constraints imposed by the Eulerian kinematic equations the **Lin constraints**. As we shall see, the diamond operation \diamond defined in equation (3.27) arises naturally in imposing the Lin constraints. Taking variations of the constrained Eulerian action,

$$\mathcal{S} = \int dt \left\{ l(\xi, a, \nu, \gamma) + \left\langle v, \frac{\partial a}{\partial t} + \mathcal{L}_\xi a \right\rangle + \left\langle \beta, \frac{\partial \gamma}{\partial t} + \mathcal{L}_\xi \gamma - d\nu - \text{ad}_\nu \gamma \right\rangle \right\} \quad (3.54)$$

yields the following **PCF Clebsch relations**,

$$\begin{aligned}
\delta\xi : \quad & \frac{\delta l}{\delta\xi} - v \diamond a - \beta \diamond \gamma = 0, \\
\delta\nu : \quad & \frac{\delta l}{\delta\nu} + d\beta - \text{ad}_\gamma^* \beta = 0, \\
\delta a : \quad & \frac{\delta l}{\delta a} - \frac{\partial v}{\partial t} - \mathcal{L}_\xi v = 0, \\
\delta v : \quad & \frac{\partial a}{\partial t} + \mathcal{L}_\xi a = 0, \\
\delta\gamma : \quad & \frac{\delta l}{\delta\gamma} - \frac{\partial\beta}{\partial t} - \mathcal{L}_\xi \beta + \text{ad}_\nu^* \beta = 0, \\
\delta\beta : \quad & \frac{\partial\gamma}{\partial t} + \mathcal{L}_\xi \gamma - d\nu - \text{ad}_\nu \gamma = 0.
\end{aligned} \tag{3.55}$$

We shall show that these Clebsch relations recover the Euler-Poincaré equations (3.23) - (3.24). (In what follows, we shall ignore boundary and endpoint terms that arise from integrating by parts.)

The **diamond operation** \diamond is defined by

$$\langle v \diamond a, \eta \rangle \equiv -\langle v, \mathcal{L}_\eta a \rangle = -\langle v, a \eta \rangle. \tag{3.56}$$

This operation is **antisymmetric**,

$$\langle v \diamond a, \eta \rangle = -\langle a \diamond v, \eta \rangle, \tag{3.57}$$

as obtained from,

$$\langle v, \mathcal{L}_\eta a \rangle + \langle \mathcal{L}_\eta v, a \rangle = 0, \quad \text{or,} \quad \langle v, a \eta \rangle + \langle v \eta, a \rangle = 0, \tag{3.58}$$

and the symmetry of the pairing $\langle \cdot, \cdot \rangle$. The diamond operation also satisfies the **chain rule under the Lie derivative**,

$$\langle \mathcal{L}_\xi (v \diamond a), \eta \rangle = \langle (\mathcal{L}_\xi v) \diamond a, \eta \rangle + \langle v \diamond (\mathcal{L}_\xi a), \eta \rangle. \tag{3.59}$$

This property can be verified, as follows,

$$\begin{aligned}
\langle \mathcal{L}_\xi v \diamond a, \eta \rangle + \langle v \diamond \mathcal{L}_\xi a, \eta \rangle &= \langle v \xi \eta, a \rangle - \langle v \eta \xi, a \rangle \\
&= \langle a, v (\text{ad}_\xi \eta) \rangle = -\langle a \diamond v, (\text{ad}_\xi \eta) \rangle \\
&= \langle \text{ad}_\xi^* (a \diamond v), \eta \rangle = \langle \mathcal{L}_\xi (a \diamond v), \eta \rangle,
\end{aligned} \tag{3.60}$$

where we have used $\langle v \xi, a \eta \rangle + \langle v \eta, a \xi \rangle = 0$, implied by (3.58), in the first step. Finally, we have the **useful identity**,

$$\langle \beta \diamond d\nu, \eta \rangle = -\langle d\beta \diamond \nu, \eta \rangle, \tag{3.61}$$

as obtained from $(d\nu)\eta = d(\nu\eta)$ and

$$\langle \beta, d(\nu\eta) \rangle + \langle d\beta, \nu\eta \rangle = 0. \tag{3.62}$$

These three properties of the \diamond operation and the PCF Clebsch relations (3.55) together imply

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \mathcal{L}_\xi\right)(v \diamond a + \beta \diamond \gamma) &= \frac{\delta l}{\delta a} \diamond a + \frac{\delta l}{\delta \nu} \diamond \nu + \frac{\delta l}{\delta \gamma_m} \diamond \gamma_m \\ &+ \left[(\text{ad}_\nu^* \beta) \diamond \gamma + \beta \diamond \text{ad}_\nu \gamma + (\text{ad}_\gamma^* \beta) \diamond \nu \right]. \end{aligned} \quad (3.63)$$

The term in square brackets is seen to vanish, upon pairing it with a vector field, integrating by parts and again using the properties of the \diamond operation. This manipulation recovers the **PCF motion equation** (3.23) as

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_\xi\right) \frac{\delta l}{\delta \xi} = \frac{\delta l}{\delta a} \diamond a + \frac{\delta l}{\delta \nu} \diamond \nu + \frac{\delta l}{\delta \gamma_m} \diamond \gamma_m, \quad (3.64)$$

since, as we have seen,

$$\mathcal{L}_\xi \frac{\delta l}{\delta \xi} = \text{ad}_\xi^* \frac{\delta l}{\delta \xi}, \quad (3.65)$$

for one-form densities such as $\delta l / \delta \xi$.

The **PCF micromotion equation** (3.24) is also recovered from the Clebsch relations (3.55). This is accomplished by taking the time derivative of the $\delta \nu$ -formula, substituting the $\delta \beta$ - and $\delta \gamma$ -formulas, and using linearity of ad^* to find

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \mathcal{L}_\xi\right) \frac{\delta l}{\delta \nu} + d \frac{\delta l}{\delta \gamma} - \text{ad}_\gamma^* \frac{\delta l}{\delta \gamma} \\ = - \left[d(\text{ad}_\nu^* \beta) + \text{ad}_{d\nu}^* \beta - \text{ad}_\gamma^* (\text{ad}_\nu^* \beta) - \text{ad}_{(\text{ad}_\nu \gamma)}^* \beta \right] \\ = - \left[\text{ad}_\nu^* d\beta - \text{ad}_\nu^* (\text{ad}_\gamma^* \beta) \right] \\ = \text{ad}_\nu^* \frac{\delta l}{\delta \nu}. \end{aligned} \quad (3.66)$$

Hence, the Clebsch relations (3.55) also recover the micromotion equation (3.24).

Remarks. From the Hamiltonian viewpoint, the pairs (v, a) and (β, γ) are canonically conjugate variables and the **Clebsch map** $(v, a, \beta, \gamma) \rightarrow (\mu, \sigma)$, with

$$\frac{\delta l}{\delta \xi} \equiv \mu = v \diamond a + \beta \diamond \gamma, \quad (3.67)$$

$$\frac{\delta l}{\delta \nu} \equiv \sigma = -d\beta + \text{ad}_\gamma^* \beta, \quad (3.68)$$

is a **Poisson map** from the canonical Poisson bracket to the Lie-Poisson Hamiltonian structure given in equation (3.46), in which $a = D$. Of course, there is no obstruction against allowing a to be any advected quantity, as discussed in Holm, Marsden and Ratiu [1998].

The generalized two-cocycle associated with the Hamiltonian matrix in (3.46) arises from the term $d\beta$ in the σ -part of this Poisson map.

Various other applications of the Lin constraint and Clebsch representation approach in formulating and analyzing ideal fluid and plasma dynamics as Hamiltonian systems appear in Holm and Kupershmidt [1983b], Marsden and Weinstein [1983], Zakharov et al. [1985], Zakharov and Kusnetsov [1997].

3.4 Conclusions for PCFs

Perfect complex fluids (PCFs) have internal variables whose micromotion is coupled to the fluid's motion. Examples of PCFs include spin-glass fluids, superfluids and liquid crystals. PCF internal variables are materially advected order parameters that may be represented equivalently as either geometrical objects, or as coset spaces of Lie groups. The new feature of PCFs relative to simple fluids with advected parameters treated in Holm, Marsden and Ratiu [1998] is the dependence of their Lagrangian

$$L : TG \times V^* \times T\mathcal{O} \longmapsto \mathbf{R},$$

on $T\mathcal{O}$, the tangent space of their order parameter group. Moreover, the diffeomorphisms G act on $T\mathcal{O}$. We treat Lagrangians that are invariant under the right actions of both the order parameter group \mathcal{O} and the diffeomorphisms G . In this case, reaching the Euler-Poincaré fluid description requires *two stages* of Lagrangian reduction,

$$(TG \times V^* \times (T\mathcal{O}/\mathcal{O}))/G \simeq \mathfrak{g} \times (V^* \times \mathfrak{o})g^{-1}(t),$$

rather than the single stage of Lagrangian reduction (with respect to the “relabeling transformations” of G) employed for simple fluids in Holm, Marsden and Ratiu [1998].

After studying the example of nematics in Section 2, we derived the Euler-Poincaré dynamics of PCFs in two stages of Lagrangian reduction in Section 3. The first stage produced the Lagrange-Poincaré equations derived from an action principle defined on the right invariant Lie algebra of the order parameter group in the Lagrangian (or material) fluid description. The second stage of Lagrangian reduction passed from the material fluid description to the Eulerian (or spatial) fluid description and produced the Euler-Poincaré equations for PCFs. We also derived these Euler-Poincaré equations using the Clebsch approach.

In addition, we used a Legendre transformation to obtain the Lie-Poisson Hamiltonian formulation of PCF dynamics in the Eulerian fluid description. The Lie-Poisson Hamiltonian formulation of these equations agreed with that found earlier in Holm and Kupershmidt [1987], Holm and Kupershmidt [1988], who treated spin-glass fluids, Yang-Mills magnetohydrodynamics and superfluid ^4He and ^3He . Thus, we found that Lagrangian reduction by stages provides a rationale for deriving this Lie-Poisson Hamiltonian formulation from the Lagrangian side. This approach also fits well with some gauge theoretical descriptions of condensed matter physics.

4 A strategy for introducing defect dynamics

Many other potential applications of the Euler-Poincaré framework abound in the physics of condensed matter. For example, besides the perfect liquid crystal dynamics treated here explicitly, the superfluid hydrodynamics of the various phases of ^3He may be treated similarly. In particular, the geometrical framework of Lagrangian reduction by stages is well-adapted to the standard identification of the phases of ^3He with the independent cosets of the order parameter group $SO(3) \times SO(3) \times U(1)$, as discussed, e.g., in Mineev [1980] and Volovick [1992]. Magnetic materials may also be treated this way. The seminal papers on the geometrical properties of magnetic materials are Dzyaloshinskii [1977], Volovik and Dotsenko [1980] and Dzyaloshinskii and Volovick [1980]. Other recent studies of the dynamics of magnetic materials and superfluid ^3He in directions relevant to the present paper appear, e.g., in Holm and Kupershmidt [1988], Balatskii [1990], Isayev and Peletminsky [1997], Isayev, Kovalevsky and Peletminsky [1997].

Most of these physical applications involve defects (imperfections, or “glitches” that appear as discontinuities in the order parameter) and one must describe their dynamics, as well. This is an area of intense investigation in many contexts in condensed matter physics. Our approach is based on an analogy with the theory of the Hall effect in a neutral multicomponent fluid plasma when inertia is negligible in one of the components (the electrons).

The Hamiltonian context for Hall effects in a neutral ion-electron plasma was considered by Holm [1987] for a normal-fluid plasma and by Holm and Kupershmidt [1987] for a multicomponent electromagnetically charged superfluid plasma. In this context, the introduction of an independent gauge field associated with the momentum of a distribution of superfluid vortex lines generalizes to apply for defects or vortices in any continuous medium possessing an order parameter description that arises from spontaneous symmetry breaking. In the remainder of this paper, we shall use the idea of reactive forces arising via the Hall effect to discuss the case of quantum vortices in superfluid Helium-II. In particular, we shall apply the Hall effect analogy to describe the hydrodynamics of the quantum vortex tangle in superfluid turbulence as an additional “third fluid” in Landau’s two-fluid fluid model. The third fluid associated with the vortex tangle carries momentum and moves with its own independent velocity in superfluid flows.

4.1 Vortices in superfluid ^4He

In superfluid ^4He the order parameter group is $U(1)$ and the defects are called vortices. These are **quantum vortices**, since their circulation comes in integer multiples of $\kappa = h/m \simeq 10^{-3} \text{cm}^2/\text{sec}$. Conservation of the number of quantum vortices moving through superfluid ^4He (and across the streamlines of the normal fluid component) is expressed by

$$\frac{d}{dt} \int_S \boldsymbol{\omega} \cdot \hat{\mathbf{n}} dS = 0, \quad (4.1)$$

where the superfluid vorticity $\boldsymbol{\omega}$ is the areal density of vortices and $\hat{\mathbf{n}}$ is the unit vector normal to the surface S whose boundary ∂S moves with the vortex line velocity \mathbf{v}_ℓ . When $\boldsymbol{\omega} = \text{curl } \mathbf{v}_s$ this is equivalent to a **vortex Kelvin theorem**

$$\frac{d}{dt} \oint_{\partial S(\mathbf{v}_\ell)} \mathbf{v}_s \cdot d\mathbf{x} = 0, \quad (4.2)$$

which in turn implies the fundamental relation

$$\frac{\partial \mathbf{v}_s}{\partial t} - \mathbf{v}_\ell \times \boldsymbol{\omega} = \nabla \mu. \quad (4.3)$$

The superfluid velocity naturally splits into $\mathbf{v}_s = \mathbf{u} - \mathbf{A}$, where $\mathbf{u} = \nabla \phi$ and (minus) the curl of \mathbf{A} yields the superfluid vorticity $\boldsymbol{\omega}$. The phase ϕ is then a regular function without singularities. This splitting will reveal that the Hamiltonian dynamics of superfluid ^4He with vortices may be expressed as an invariant subsystem of a larger Hamiltonian system in which \mathbf{u} and \mathbf{A} have independent evolution equations. We begin by defining a phase frequency in the normal velocity frame as

$$\frac{\partial \phi}{\partial t} + \mathbf{v}_n \cdot \nabla \phi = \nu. \quad (4.4)$$

The mass density ρ and the phase ϕ are canonically conjugate in the Hamiltonian formulation of the Landau two-fluid model. Therefore, $\nu = -\delta h/\delta\rho$ for a given Hamiltonian h and $\mathbf{u} = \nabla\phi$ satisfies

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{v}_n \cdot \nabla \mathbf{u} + (\nabla \mathbf{v}_n)^T \cdot \mathbf{u} = -\nabla \frac{\delta h}{\delta \rho}. \quad (4.5)$$

The mass density ρ satisfies the dual equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}_n) = -\nabla \cdot \frac{\delta h}{\delta \mathbf{u}}. \quad (4.6)$$

In the last two equations, we see the expected two-cocycle terms for Landau's perfect superfluid, in which $\boldsymbol{\omega} = 0$. Here \mathbf{u} and ρ in the superfluid play the roles of γ and σ for the PCFs. The curvature $\boldsymbol{\omega}$ is nonvanishing now, because of the vortices (defects) that are represented by \mathbf{A} .

Perhaps not surprisingly, the rotational and potential components of the superfluid velocity must satisfy similar equations, but the rotational component must be advected by another velocity – the vortex line velocity \mathbf{v}_ℓ – instead of the normal velocity \mathbf{v}_n that advects \mathbf{u} . Absorbing all gradients into \mathbf{u} yields the form of the equation we should expect for \mathbf{A} ,

$$\frac{\partial \mathbf{A}}{\partial t} + \mathbf{v}_\ell \times \boldsymbol{\omega} = 0. \quad (4.7)$$

Taking the difference of the equations for \mathbf{u} and \mathbf{A} then recovers equation (4.3) as

$$\frac{\partial \mathbf{v}_s}{\partial t} - \mathbf{v}_\ell \times \boldsymbol{\omega} = -\nabla \left(\mathbf{v}_n \cdot \mathbf{u} + \frac{\delta h}{\delta \rho} \right) \quad \text{with} \quad \mathbf{v}_s = \mathbf{u} - \mathbf{A}, \quad (4.8)$$

in which regularity of the phase ϕ allows one to set $\text{curl } \mathbf{u} = 0$. It remains to determine \mathbf{v}_ℓ from the Euler-Poincaré formulation. Including the additional degree of freedom \mathbf{A} allows the vortex lines to move relative to both the normal and super components of the fluid, and thereby introduces additional reactive forces associated with the momentum of the vortex lines.

For superfluid ^4He with vortices, the momenta conjugate to the velocities \mathbf{v}_n and \mathbf{v}_ℓ shall be our basic dynamical variables. To develop the Euler-Poincaré formulation of this problem, we must consider a Lagrangian that first of all is invariant under the order parameter group $\mathcal{O} = U(1)$. The Lagrangian must also be invariant under *two types of diffeomorphisms*: one corresponding to the material motion of the normal fluid G_n and another corresponding to the motion of the vortices G_ℓ . Thus we consider a Lagrangian that allows the following *direct product* of group reductions,

$$\begin{aligned} \left((TG_n \times V^* \times (T\mathcal{O}/\mathcal{O})) / G_n \right) &\times \left((TG_\ell \times V^*) / G_\ell \right) \\ &\simeq \left(\mathfrak{g}_n \times (V^* \times \mathfrak{o}) g_n^{-1}(t) \right) \times \left(\mathfrak{g}_\ell \times V^* g_\ell^{-1}(t) \right). \end{aligned}$$

We denote the corresponding dependence in this Lagrangian as

$$l(\mathbf{v}_n, S, \nu, \mathbf{u}; \mathbf{v}_\ell, n). \quad (4.9)$$

Here \mathfrak{g}_n and \mathfrak{g}_ℓ denote the Lie algebras of vector fields associated to the velocities \mathbf{v}_n and \mathbf{v}_ℓ , respectively. Also \mathfrak{o} denotes the Lie algebra of the Abelian gauge group $U(1)$; so \mathfrak{o} contains ν and \mathbf{u} . The V^* in each factor denotes the corresponding advected densities: entropy S advected

by \mathbf{v}_n and vortex inertial mass n advected by \mathbf{v}_ℓ . We denote the inverse right actions of the two diffeomorphisms as $g_n^{-1}(t)$ and $g_\ell^{-1}(t)$. At a given time t , these actions separately map spatial variables back to coordinates moving with the normal material and with the vortices, respectively.

According to the Euler-Poincaré action principle, the following dynamical equations are generated by this Lagrangian, cf. (3.23) and (3.24)

$$\begin{aligned}\frac{\partial}{\partial t} \frac{\delta l}{\delta v_n} &= -\text{ad}_{v_n}^* \frac{\delta l}{\delta v_n} + \frac{\delta l}{\delta S} \diamond S + \frac{\delta l}{\delta \nu} \diamond \nu + \frac{\delta l}{\delta u_m} \diamond u_m, \\ \frac{\partial}{\partial t} \frac{\delta l}{\delta \nu} &= -\text{div} \left(\mathbf{v}_n \frac{\delta l}{\delta \nu} + \frac{\delta l}{\delta \mathbf{u}} \right), \\ \frac{\partial}{\partial t} \frac{\delta l}{\delta v_\ell} &= -\text{ad}_{v_\ell}^* \frac{\delta l}{\delta v_\ell} + \frac{\delta l}{\delta n} \diamond n.\end{aligned}\tag{4.10}$$

These are the **Euler-Poincaré equations** for a superfluid with vortices. The Eulerian kinematic equations are, cf. (2.61-2.62)

$$\begin{aligned}\frac{\partial S}{\partial t} &= -\text{div}(S \mathbf{v}_n), \\ \frac{\partial \mathbf{u}}{\partial t} &= \mathbf{v}_n \times \text{curl } \mathbf{u} - \nabla (\mathbf{u} \cdot \mathbf{v}_n - \nu), \\ \frac{\partial n}{\partial t} &= -\text{div}(n \mathbf{v}_\ell).\end{aligned}$$

If \mathbf{v}_ℓ and n are absent, these equations reduce to the equations for a PCF with broken $U(1)$ symmetry. The momentum density conjugate to the frequency ν is the total mass density given by $\rho = -\delta l / \delta \nu$, which satisfies the equation above. So far, this is interesting, but standard in the present context. However, now a new feature develops because of the physical description of superfluids. Physically, nothing is known on the Lagrangian side about the relation of the gauge frequency ν to the other variables. However, on the Hamiltonian side we know from the Legendre transformation that $\nu = -\delta h / \delta \rho$. Moreover, the thermodynamic energy on the Hamiltonian side is a known function of ρ . This means we should leave the Lagrangian side to finish determining the dynamics for superfluid ^4He with vortices. The following Hamiltonian description for this dynamics is derived in Holm [2000].

Proposition: *The dynamics for superfluid ^4He with vortices follows from a Lie-Poisson bracket whose Hamiltonian matrix separates into two pieces given by*⁷

$$\frac{\partial}{\partial t} \begin{bmatrix} M_i \\ S \\ \rho \\ u_i \end{bmatrix} = - \begin{bmatrix} M_j \partial_i + \partial_j M_i & S \partial_i & \rho \partial_i & \partial_j u_i - u_{j,i} \\ \partial_j S & 0 & 0 & 0 \\ \partial_j \rho & 0 & 0 & \partial_j \\ u_j \partial_i + u_{i,j} & 0 & \partial_i & 0 \end{bmatrix} \begin{bmatrix} \delta h / \delta M_j \\ \delta h / \delta S \\ \delta h / \delta \rho \\ \delta h / \delta u_j \end{bmatrix}, \tag{4.11}$$

and

$$\frac{\partial}{\partial t} \begin{bmatrix} N_i \\ n \end{bmatrix} = - \begin{bmatrix} N_j \partial_i + \partial_j N_i & n \partial_i \\ \partial_j n & 0 \end{bmatrix} \begin{bmatrix} \delta h / \delta N_j \\ \delta h / \delta n \end{bmatrix}, \tag{4.12}$$

⁷The first of these is the Hamiltonian matrix for a PCF with broken $U(1)$ symmetry. The other Hamiltonian matrix gives the standard semidirect-product Lie-Poisson bracket, without two-cocycles. This combination of Hamiltonian matrices was first introduced in Holm and Kupershmidt [1987] for superfluid plasmas.

where $M = \delta l / \delta v_n$, $N = \delta l / \delta v_\ell$, $\rho = -\delta l / \delta v$ and the Hamiltonian h is the Legendre transform of the Lagrangian l in (4.9) with respect to v_n , v_ℓ and v .

Corollary: *If the Hamiltonian has no explicit spatial dependence, then total momentum conservation holds as,*

$$\frac{\partial}{\partial t}(M_j + N_j) = \{M_j + N_j, h\} = -\frac{\partial}{\partial x^k} T_j^k.$$

Suppose the Hamiltonian density has dependence $h(\mathbf{M}, \rho, S, n, \mathbf{v}_s, \boldsymbol{\omega}, \mathbf{A})$, in which

$$\mathbf{v}_s = \mathbf{u} - \mathbf{A}, \quad \mathbf{A} = -\mathbf{N}/n \quad \text{and} \quad \boldsymbol{\omega} = \text{curl } \mathbf{v}_s.$$

Then, the stress tensor T_j^k is expressed in terms of derivatives of the Hamiltonian as

$$T_j^k = M_j \frac{\partial h}{\partial M_k} + v_{sj} \left(\frac{\partial h}{\partial v_{sk}} + \left(\text{curl } \frac{\partial h}{\partial \boldsymbol{\omega}} \right)_k \right) - v_{sl,j} \epsilon_{mlk} \frac{\partial h}{\partial \omega_m} + \delta_j^k P - A_j \frac{\partial h}{\partial A_k} \Big|_{\mathbf{v}_s}.$$

where the pressure P is defined as

$$P = M_l \frac{\partial h}{\partial M_l} + \rho \frac{\partial h}{\partial \rho} + S \frac{\partial h}{\partial S} + n \frac{\partial h}{\partial n} - h.$$

Thus, the dependence of the Hamiltonian h on the vorticity $\boldsymbol{\omega}$ introduces reactive stresses due to the motion of the vortices.

Details of the choice of Hamiltonian, as well as the derivation and interpretation of the explicit equations are given in Holm [2000]. Other results include the transformation of these equations to a rotating reference frame and the resulting Taylor-Proudman theorem for superfluid ^4He with vortices. The generalization of this idea to complex fluids with nonabelian broken symmetries will be discussed elsewhere.

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Appendix. External torques and partial Lagrangian reduction.

The anisotropic dielectric and diamagnetic effects on the director angular momentum due to external electric and magnetic fields can be restored by adding the torques from equation (2.38) to the right hand side of the second equation in (2.52). Knowing they can be restored this way, one could simply ignore the external torques. However, the major applications of liquid crystals involve these torques and their restoration also provides an example of *partial Lagrangian reduction*. This example also briefly recapitulates the procedure of Lagrangian reduction by stages used in the remainder of the paper.

Partially reduced Lagrange-Poincaré equations. Restoring the effects of external torques requires that we consider Hamilton's principle for a Lagrangian that still retains its dependence on the director \mathbf{n} ,

$$\mathcal{S}' = \int dt \int d^3X \mathcal{L}(\dot{\mathbf{x}}, J, \mathbf{n}, \boldsymbol{\nu}, \nabla \mathbf{n}). \quad (4.13)$$

In this case, the partial reduction of the Lagrangian dependence from $\dot{\mathbf{n}}$ to $\boldsymbol{\nu} = \mathbf{n} \times \dot{\mathbf{n}}$ proceeds as follows. The variation of $\boldsymbol{\nu} = \mathbf{n} \times \dot{\mathbf{n}}$ gives

$$\delta \boldsymbol{\nu} = (\mathbf{n} \times \delta \mathbf{n})^\cdot - 2\dot{\mathbf{n}} \times \delta \mathbf{n} = (\mathbf{n} \times \delta \mathbf{n})^\cdot - 2\boldsymbol{\nu} \times (\mathbf{n} \times \delta \mathbf{n}). \quad (4.14)$$

Moreover, we have the relation

$$\delta \mathbf{n} \cdot \mathbf{A} = (\mathbf{n} \times \delta \mathbf{n}) \cdot (\mathbf{n} \times \mathbf{A}) = |\mathbf{n}|^2 \delta \mathbf{n} \cdot \mathbf{A} - (\mathbf{n} \cdot \delta \mathbf{n})(\mathbf{n} \cdot \mathbf{A}), \quad (4.15)$$

for any vector \mathbf{A} , since $|\mathbf{n}|^2 = 1$, which implies that $\mathbf{n} \cdot \delta \mathbf{n} = 0$. Substituting the identity (4.14) for $\delta \boldsymbol{\nu}$ and the relation (4.15) into Hamilton's principle implies the following replacements,

$$\delta \mathbf{n} \cdot \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{n}}} \right)^\cdot \implies (\mathbf{n} \times \delta \mathbf{n}) \cdot \left[\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\nu}} \right)^\cdot - 2\boldsymbol{\nu} \times \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nu}} \right] \quad (4.16)$$

$$\delta \mathbf{n} \cdot \left(\frac{\partial \mathcal{L}}{\partial \mathbf{n}} \right) \implies (\mathbf{n} \times \delta \mathbf{n}) \cdot \left(\mathbf{n} \times \frac{\partial \mathcal{L}}{\partial \mathbf{n}} \right) \quad (4.17)$$

Varying the action \mathcal{S}' in the fields \mathbf{x} , \mathbf{n} and $\boldsymbol{\nu}$ at fixed material position \mathbf{X} and time t now gives

$$\begin{aligned} \delta \mathcal{S}' = - \int dt \int d^3X \Big\{ & \delta x_p \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_p} \right)^\cdot + J \frac{\partial}{\partial x_p} \frac{\partial \mathcal{L}}{\partial J} - J \frac{\partial}{\partial x_m} \left(J^{-1} \frac{\partial \mathcal{L}}{\partial \mathbf{n}_{,m}} \cdot \mathbf{n}_{,p} \right) \right] \\ & + (\mathbf{n} \times \delta \mathbf{n}) \cdot \left[\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\nu}} \right)^\cdot - 2\boldsymbol{\nu} \times \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nu}} \right] \\ & - (\mathbf{n} \times \delta \mathbf{n}) \cdot \left[\mathbf{n} \times \left(\frac{\partial \mathcal{L}}{\partial \mathbf{n}} - J \frac{\partial}{\partial x_m} \left(J^{-1} \frac{\partial \mathcal{L}}{\partial \mathbf{n}_{,m}} \right) \right) \right] \Big\}, \end{aligned} \quad (4.18)$$

with the same natural (homogeneous) boundary conditions as before. Consequently, the action principle $\delta \mathcal{S}' = 0$ yields the following **partially reduced Lagrange-Poincaré equations** for liquid crystals,

$$\begin{aligned} \delta x_p : \quad & \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_p} \right)^\cdot + J \frac{\partial}{\partial x_p} \frac{\partial \mathcal{L}}{\partial J} - J \frac{\partial}{\partial x_m} \left(J^{-1} \frac{\partial \mathcal{L}}{\partial \mathbf{n}_{,m}} \cdot \mathbf{n}_{,p} \right) = 0, \\ \mathbf{n} \times \delta \mathbf{n} : \quad & \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\nu}} \right)^\cdot - 2\boldsymbol{\nu} \times \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nu}} - \mathbf{n} \times \left(\frac{\partial \mathcal{L}}{\partial \mathbf{n}} - J \frac{\partial}{\partial x_m} \left(J^{-1} \frac{\partial \mathcal{L}}{\partial \mathbf{n}_{,m}} \right) \right) = 0. \end{aligned} \quad (4.19)$$

One may compare these with the more completely reduced Lagrange-Poincaré equations for liquid crystals in equations (2.52). The explicit dependence of the Lagrangian on the director \mathbf{n} introduces torques and stresses not seen for Lagrangians depending only on $\boldsymbol{\nu}$ and $\boldsymbol{\gamma} = \mathbf{n} \times d\mathbf{n}$.

Partially reduced Euler-Poincaré equations. We are dealing with Hamilton's principle for a Lagrangian in the class

$$\mathcal{S}' = \int dt \int d^3x \ell(\mathbf{u}, D, \mathbf{n}, \boldsymbol{\nu}, \nabla \mathbf{n}), \quad (4.20)$$

in terms of the Lagrangian density ℓ given by

$$\ell(\mathbf{u}, D, \mathbf{n}, \boldsymbol{\nu}, \nabla \mathbf{n}) d^3x = \mathcal{L}\left(\dot{x}g^{-1}(t), Jg^{-1}(t), \mathbf{n}g^{-1}(t), \boldsymbol{\nu}g^{-1}(t), \nabla \mathbf{n}g^{-1}(t)\right) \left(d^3Xg^{-1}(t)\right). \quad (4.21)$$

The variations of the Eulerian fluid quantities are computed from their definitions to be,

$$\delta u_j = \frac{\partial \eta_j}{\partial t} + u_k \frac{\partial \eta_j}{\partial x_k} - \eta_k \frac{\partial u_j}{\partial x_k}, \quad (4.22)$$

$$\delta D = -\frac{\partial D \eta_j}{\partial x_j}, \quad (4.23)$$

$$\delta \boldsymbol{\nu} = \frac{\partial \boldsymbol{\Sigma}}{\partial t} + u_m \frac{\partial \boldsymbol{\Sigma}}{\partial x_m} - 2\boldsymbol{\nu} \times \boldsymbol{\Sigma} - \eta_m \frac{\partial \boldsymbol{\nu}}{\partial x_m}, \quad (4.24)$$

$$\delta \mathbf{n} = -\mathbf{n} \times \boldsymbol{\Sigma} - \eta_k \frac{\partial \mathbf{n}}{\partial x_k}, \quad (4.25)$$

where $\boldsymbol{\Sigma}(x, t) \equiv (\mathbf{n} \times \delta \mathbf{n})(X, t)g^{-1}(t)$ and $\eta \equiv \delta g g^{-1}(t)$.

We compute the variation of the action (4.20) in Eulerian variables at fixed time t and spatial position \mathbf{x} as, cf. equation (2.65),

$$\begin{aligned} \delta \mathcal{S} &= \int dt \int d^3x \left[\frac{\delta \ell}{\delta u_j} \delta u_j + \frac{\delta \ell}{\delta D} \delta D + \frac{\delta \ell}{\delta \boldsymbol{\nu}} \cdot \delta \boldsymbol{\nu} + \frac{\delta \ell}{\delta \mathbf{n}} \cdot \delta \mathbf{n} \right] \\ &= \int dt \int d^3x \left\{ \eta_j \left[-\frac{\partial}{\partial t} \frac{\delta \ell}{\delta u_j} - \frac{\delta \ell}{\delta u_k} \frac{\partial u_k}{\partial x_j} - \frac{\partial}{\partial x_k} \left(\frac{\delta \ell}{\delta u_j} u_k \right) \right. \right. \\ &\quad \left. \left. + D \frac{\partial}{\partial x_j} \frac{\delta \ell}{\delta D} - \frac{\delta \ell}{\delta \boldsymbol{\nu}} \cdot \frac{\partial \boldsymbol{\nu}}{\partial x_j} - \frac{\delta \ell}{\delta \mathbf{n}} \cdot \frac{\partial \mathbf{n}}{\partial x_j} \right] \right. \\ &\quad \left. + \boldsymbol{\Sigma} \cdot \left[-\frac{\partial}{\partial t} \frac{\delta \ell}{\delta \boldsymbol{\nu}} - \frac{\partial}{\partial x_m} \left(\frac{\delta \ell}{\delta \boldsymbol{\nu}} u_m \right) + 2\boldsymbol{\nu} \times \frac{\delta \ell}{\delta \boldsymbol{\nu}} + \mathbf{n} \times \frac{\delta \ell}{\delta \mathbf{n}} \right] \right. \\ &\quad \left. + \frac{\partial}{\partial t} \left[\eta_j \frac{\delta \ell}{\delta u_j} + \boldsymbol{\Sigma} \cdot \frac{\delta \ell}{\delta \boldsymbol{\nu}} \right] \right. \\ &\quad \left. + \frac{\partial}{\partial x_m} \left[\eta_j \left(\frac{\delta \ell}{\delta u_j} u_m - D \frac{\delta \ell}{\delta D} \delta_{jm} \right) + \boldsymbol{\Sigma} \cdot \left(\frac{\delta \ell}{\delta \boldsymbol{\nu}} u_m \right) \right] \right\}, \end{aligned} \quad (4.26)$$

where we have substituted the variational expressions (4.22)-(4.25) and integrated by parts. Hence, we obtain the **partially reduced Euler-Poincaré equations** for liquid crystals,

$$\eta_j : \quad \frac{\partial}{\partial t} \frac{\delta \ell}{\delta u_j} = -\frac{\delta \ell}{\delta u_k} \frac{\partial u_k}{\partial x_j} - \frac{\partial}{\partial x_k} \left(\frac{\delta \ell}{\delta u_j} u_k \right) + D \frac{\partial}{\partial x_j} \frac{\delta \ell}{\delta D} - \frac{\delta \ell}{\delta \boldsymbol{\nu}} \cdot \frac{\partial \boldsymbol{\nu}}{\partial x_j} - \frac{\delta \ell}{\delta \mathbf{n}} \cdot \frac{\partial \mathbf{n}}{\partial x_j}, \quad (4.27)$$

$$\boldsymbol{\Sigma} : \quad \frac{\partial}{\partial t} \frac{\delta \ell}{\delta \boldsymbol{\nu}} = -\frac{\partial}{\partial x_m} \left(\frac{\delta \ell}{\delta \boldsymbol{\nu}} u_m \right) + 2\boldsymbol{\nu} \times \frac{\delta \ell}{\delta \boldsymbol{\nu}} + \mathbf{n} \times \frac{\delta \ell}{\delta \mathbf{n}}. \quad (4.28)$$

One may compare these with the more completely reduced Euler-Poincaré equations for liquid crystals in equations (2.66) and (2.67). Again one sees the torques and stresses generated by the explicit dependence of the Lagrangian on the director \mathbf{n} .

Partially reduced Hamiltonian dynamics of liquid crystals The Euler-Lagrange-Poincaré formulation of liquid crystal dynamics obtained so far allows passage to the corresponding Hamiltonian formulation via the following **Legendre transformation** of the partially reduced Lagrangian ℓ in the velocities \mathbf{u} and $\boldsymbol{\nu}$, in the Eulerian fluid description,

$$m_i = \frac{\delta \ell}{\delta u_i}, \quad \boldsymbol{\sigma} = \frac{\delta \ell}{\delta \boldsymbol{\nu}}, \quad h(\mathbf{m}, D, \boldsymbol{\sigma}, \mathbf{n}) = m_i u_i + \boldsymbol{\sigma} \cdot \boldsymbol{\nu} - \ell(\mathbf{u}, D, \boldsymbol{\nu}, \mathbf{n}). \quad (4.29)$$

Accordingly, one computes the derivatives of h as

$$\frac{\delta h}{\delta m_i} = u_i, \quad \frac{\delta h}{\delta \boldsymbol{\sigma}} = \boldsymbol{\nu}, \quad \frac{\delta h}{\delta D} = -\frac{\delta \ell}{\delta D}, \quad \frac{\delta h}{\delta \mathbf{n}} = -\frac{\delta \ell}{\delta \mathbf{n}}. \quad (4.30)$$

Consequently, the Euler-Poincaré equations (2.66) - (4.28) and the auxiliary kinematic equations (2.61) - (2.62) for liquid crystal dynamics in the Eulerian description imply the following equations, for the Legendre-transformed variables, $(\mathbf{m}, D, \boldsymbol{\sigma}, \mathbf{n})$,

$$\begin{aligned} \frac{\partial m_i}{\partial t} = & -m_j \frac{\partial}{\partial x_i} \frac{\delta h}{\delta m_j} - \frac{\partial}{\partial x_j} \left(m_i \frac{\delta h}{\delta m_j} \right) - D \frac{\partial}{\partial x_i} \frac{\delta h}{\delta D} \\ & + \left(\frac{\partial \mathbf{n}}{\partial x_i} \right) \cdot \frac{\delta h}{\delta \mathbf{n}} - \boldsymbol{\sigma} \cdot \frac{\partial}{\partial x_i} \frac{\delta h}{\delta \boldsymbol{\sigma}}, \end{aligned} \quad (4.31)$$

$$\frac{\partial D}{\partial t} = -\frac{\partial}{\partial x_j} \left(D \frac{\delta h}{\delta m_j} \right), \quad (4.32)$$

$$\frac{\partial \mathbf{n}}{\partial t} = -\left(\frac{\partial \mathbf{n}}{\partial x_j} \right) \frac{\delta h}{\delta m_j} - \mathbf{n} \times \frac{\delta h}{\delta \boldsymbol{\sigma}} \quad (4.33)$$

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} = -\frac{\partial}{\partial x_j} \left(\boldsymbol{\sigma} \frac{\delta h}{\delta m_j} \right) - \mathbf{n} \times \frac{\delta h}{\delta \mathbf{n}} - 2\boldsymbol{\sigma} \times \frac{\delta h}{\delta \boldsymbol{\sigma}}. \quad (4.34)$$

These equations are **Hamiltonian**. Assembling the liquid crystal equations (4.31) - (4.34) into the Hamiltonian form (2.79) gives,

$$\frac{\partial}{\partial t} \begin{bmatrix} m_i \\ D \\ \mathbf{n} \\ \boldsymbol{\sigma} \end{bmatrix} = - \begin{bmatrix} m_j \partial_i + \partial_j m_i & D \partial_i & -\mathbf{n}_{,i} \cdot & \boldsymbol{\sigma} \cdot \partial_i \\ \partial_j D & 0 & 0 & 0 \\ \mathbf{n}_{,j} & 0 & 0 & \mathbf{n} \times \\ \partial_j \boldsymbol{\sigma} & 0 & \mathbf{n} \times & 2\boldsymbol{\sigma} \times \end{bmatrix} \begin{bmatrix} \delta h / \delta m_j \\ \delta h / \delta D \\ \delta h / \delta \mathbf{n} \\ \delta h / \delta \boldsymbol{\sigma} \end{bmatrix} \quad (4.35)$$

One may compare this with the more completely reduced Hamiltonian matrix form for liquid crystals in equations (2.81). The Jacobi identity for the Poisson bracket defined by this Hamiltonian matrix is guaranteed by associating it to the dual of a semidirect-product Lie algebra that, in this case, has no two-cocycles. The two-cocycles are generated by the further transformation to $\boldsymbol{\gamma} = \mathbf{n} \times d\mathbf{n}$.

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